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A Simple Two-Factor European Call Pricing

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Agenda

I- Motivations and setting

II- Theoretical framework

III- Option pricing and Greeks

IV- Empirical application

V- Concluding remarks

I – Motivations: Theoretical & Empirical facts

- Basic assumptions of Black & Scholes (73) world are violated (Bekaert & Wu [2000], Black [76], Schwert [89,98]).
- Non-normality and asymmetry in asset returns are observed.

$\Rightarrow$ Systematic valuation biases (time-varying volatility / volatility as a function of moneyness).

- The shorter the maturity of options, the higher the biases (Brown & Robinson [2002]).

$\Rightarrow$ Higher moments of the distribution of the underlying stock's terminal price allow for reducing those biases (Brown & Robinson [2002], Corrado & Su [96,97]).

I – Our viewpoint

- Asymmetry in asset returns comes from the combination of or tradeoff between two main sources of risk.
- Asset return & Sharpe (63) $\Rightarrow$ systematic risk factor component and idiosyncratic risk factor component.
- There is a wide literature about such components:
  - Daves *et al*. (2000);
  - Malkiel & Xu (2002, 2003);
  - Campbell *et al*. (2001).

II – Basic framework

- Any stock's evolution depends on a systematic factor and an idiosyncratic factor of risk, which are independent.
- Let S_t^i, M_t, I_t^i and β_i be respectively the current price of stock i, current systematic and idiosyncratic risk factor levels, and finally the stock's beta (Sharpe [63,64]).
- Under specific constraints in the historical universe, we set:

$$S_t^i = M_t^{\beta_i} I_t^i$$

$$\frac{dM_t}{M_t} = \mu_M \, dt + \sigma_M \, dW_t$$

$$\frac{dI_t^i}{I_t^i} = \mu_{I^i} \, dt + \sigma_{I^i} \, dW_t^i$$

II – Combination of risks & risk level

- Finally, we get the global dynamic of stock i in the historical universe:

$$\frac{dS_t^i}{S_t^i} = \mu_{S^i}\, dt + \sigma_{S^i}\, d\omega_t^i$$

$$\mu_{S^i} = \beta_i \mu_M + \mu_{I^i} + \frac{1}{2}\beta_i\left(\beta_i - 1\right)\sigma_M^2$$

$$\sigma_{S^i}^2 = \beta_i^2 \sigma_M^2 + \sigma_{I^i}^2$$

$$d\omega_t^i = \frac{\beta_i \sigma_M\, dW_t + \sigma_{I^i}\, dW_t^i}{\sigma_{S^i}}$$

II – Risk-neutral universe (1/2)

- Hence, the stock's global dynamic in the risk-neutral universe rewrites:

$$d\ln\left(S_t^i\right) = \left(\mu_{S^i}^* - \frac{\sigma_{S^i}^2}{2}\right)dt + \sigma_{S^i}\, d\omega_t^{i*}$$

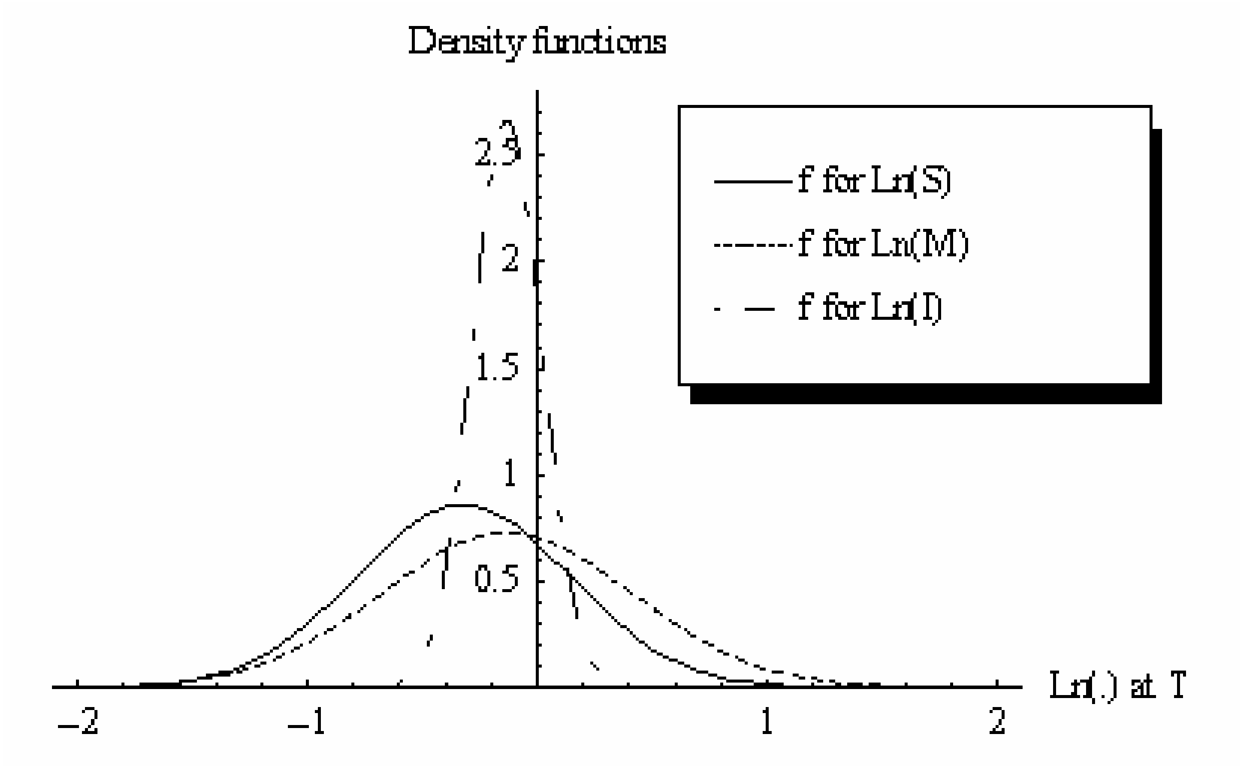
$$\mu_{S^i}^* = r\left(\beta_i + 1\right) + \frac{1}{2}\beta_i\left(\beta_i - 1\right)\sigma_M^2$$

$$d\ln\left(M_t\right) = \left(r - \frac{\sigma_M^2}{2}\right)dt + \sigma_M\, dW_t^*$$

$$d\ln\left(I_t^i\right) = \left(r - \frac{\sigma_{I^i}^2}{2}\right)dt + \sigma_{I^i}\, dW_t^{i*}$$

II – Risk-neutral universe (2/2)

- ***Risk tradeoff***: Risk-neutral densities at maturity T with r=8%, T-t=0.25 year, M_t=1, I_t^i=0.8, σ_M=1.1, σ_{I^i}=0.3, and β_i=0.8.



III – European call pricing

- All the assumptions of the BS (73) world hold here.
- A European call on stock i with strike K, maturing in $\tau=T-t$ year(s) is priced as $C(\tau, S_t^i, M_t)=E^Q[e^{-r\tau} Max(0,S_T^i-K)/F_t]$ under risk-neutral probability Q.
- Our closed-form pricing formula writes:

$$C(\tau,S_t^i,M_t)\equiv C(\tau,S_t^i,r,K,\sigma_{S^i},\sigma_M,\beta_i)=S_t^i\, e^{u\tau}N(d_1)-K e^{-r\tau}N(d_2)$$

$$u\equiv u(r,\sigma_M,\beta_i)=\mu^*_{S^i}-r=r\beta_i+\frac{1}{2}\beta_i(\beta_i-1)\sigma_M^2$$

$$d_1\equiv d_1(\tau,S_t^i,r,K,\sigma_{S^i},\sigma_M,\beta_i)=\frac{\ln\left(\frac{S_t^i}{K}\right)+\left(\mu^*_{S^i}+\frac{\sigma^2_{S^i}}{2}\right)\tau}{\sigma_{S^i}\sqrt{\tau}}$$

$$d_2\equiv d_2(\tau,S_t^i,r,K,\sigma_{S^i},\sigma_M,\beta_i)=d_1-\sigma_{S^i}\sqrt{\tau}=\frac{\ln\left(\frac{S_t^i}{K}\right)+\left(\mu^*_{S^i}-\frac{\sigma^2_{S^i}}{2}\right)\tau}{\sigma_{S^i}\sqrt{\tau}}$$

III - Greeks

- The sensitivity of our European call pricing to key parameters is as follows:

$$\Delta = \frac{\partial C\left(\tau, S_t^i, r, K, \sigma_{S^i}, \sigma_M, \beta_i\right)}{\partial S_t^i} = e^{u\tau} N\left(d_1\right) > 0$$

$$\theta = \frac{\partial C\left(\tau, S_t^i, r, K, \sigma_{S^i}, \sigma_M, \beta_i\right)}{\partial \tau} = -\frac{\partial C\left(\tau, S_t^i, r, K, \sigma_{S^i}, \sigma_M, \beta_i\right)}{\partial t} = S_t^i \, u \, e^{u\tau} N\left(d_1\right) + K \, r \, e^{-r\tau} N\left(d_2\right) + S_t^i \, e^{u\tau} \, n\left(d_1\right) \frac{\sigma_{S^i}}{2\sqrt{\tau}}$$

$$\rho = \frac{\partial C\left(\tau, S_t^i, r, K, \sigma_{S^i}, \sigma_M, \beta_i\right)}{\partial r} = S_t^i \, \beta_i \, \tau e^{u\tau} N\left(d_1\right) + K \, \tau e^{-r\tau} N\left(d_2\right)$$

$$\upsilon_{S_t^i} = \frac{\partial C\left(\tau, S_t^i, r, K, \sigma_{S^i}, \sigma_M, \beta_i\right)}{\partial \sigma_{S^i}} = S_t^i \, e^{u\tau} \, n\left(d_1\right) \sqrt{\tau} > 0$$

$$\upsilon_M = \frac{\partial C\left(\tau, S_t^i, r, K, \sigma_{S^i}, \sigma_M, \beta_i\right)}{\partial \sigma_M} = S_t^i \, \beta_i\left(\beta_i - 1\right) \sigma_M \, \tau e^{u\tau} N\left(d_1\right) = S_t^i \, \beta_i\left(\beta_i - 1\right) \sigma_M \, \tau \, \Delta$$

$$\beta = \frac{\partial C\left(\tau, S_t^i, r, K, \sigma_{S^i}, \sigma_M, \beta_i\right)}{\partial \beta_i} = S_t^i \left(\beta_i \, \sigma_M^2 - \frac{\sigma_M^2}{2} + r\right) \tau e^{u\tau} N\left(d_1\right)$$

IV - Data

- Daily data from 2nd January to 26th March 2002 (60 observations per series).
- Closing prices of 10 French stocks + 1 French stock index (CAC40 Index).
- Related European calls with different strikes.
- Options' maturity: 27th March 2002 (three-month European stock calls).
- No dividend paid over our time horizon.
- One-, two-, and three-month French risk free rates.

$\Rightarrow$ Quadratic interpolation of French risk free rates to match European calls' time to maturity.

IV – Implied parameters

- Unobserved key parameters (to be estimated):
  - ◊ Two-factor pricing: σ_{S^i}, σ_M, and β_i.
  - ◊ BS (73) pricing: σ_{S^i}.
- Quadratic minimization problem at each time t:
  - ◊ BS (73): 1 step

$$\underset{\sigma_{S^i}}{Min}\left\{\sum_{j=1}^{n_K^i}\left[C^*\left(\tau,S_t^i,r,K_j^i,\sigma_{S^i}\right)-\hat{C}\left(\tau,K_j^i,S_t\right)\right]^2\right\}$$

  - ◊ Two-factor pricing: 2 steps

$$\underset{\sigma_{S^i},\,\sigma_M,\,\beta_i}{Min}\left\{\sum_{j=1}^{n_K^i}\left[C\left(\tau,S_t^i,r,K_j^i,\sigma_{S^i},\sigma_M,\beta_i\right)-\hat{C}\left(\tau,K_j^i,S_t\right)\right]^2\right\}$$

$$\underset{\sigma_{S^i},\,\beta_i}{Min}\left\{\sum_{j=1}^{n_K^i}\left[C\left(\tau,S_t^i,r,K_j^i,\sigma_{S^i},\overline{\sigma}_M,\beta_i\right)-\hat{C}\left(\tau,K_j^i,S_t\right)\right]^2\right\}$$

⇒ Time varying implied parameters!

IV – Implied parameters estimates

- Statistics about implied global volatilities (two-factor vs. *BS*):

Two-factor

Asset	Mean	Stand. dev.(%)	Skewness	Excess kurtosis
$\bar{\sigma}_M$	0.2200	3.7594	0.2662	-0.7698
Ai	0.2692	1.7297	-0.4398	-0.7834
Bn	0.2512	2.5880	-1.1002	1.0818
Or	0.2885	2.2044	0.5296	0.0837
Rno	0.3645	2.9632	0.4677	-1.3760
Su	0.4193	1.6247	-0.6971	0.8415
Gle	0.3094	1.5521	0.0458	-0.1452
Tmm	0.4855	2.0368	0.3692	-0.6374
Fp	0.2558	1.8416	-1.1227	0.6810
Fr	0.3431	1.6222	-0.7786	-0.7979
Ex	0.3888	2.1026	-0.1159	-0.5291

BS (73)

Asset	Mean	Stand. dev.(%)	Skewness	Excess kurtosis
Cac40	0.2451	1.0442	-0.2758	-0.7163
Ai	0.2769	1.6186	-0.7654	0.2546
Bn	0.2580	2.5910	-1.3836	1.3068
Or	0.3006	2.0969	-0.1083	-0.2674
Rno	0.3742	2.9084	0.3397	-1.4892
Su	0.4270	1.7575	-0.5852	0.2761
Gle	0.3167	1.6490	0.2277	-0.6381
Tmm	0.4970	1.9119	0.3365	-0.8434
Fp	0.2680	1.3109	-1.0534	0.9042
Fr	0.3518	1.7888	-0.7247	-0.8511
Ex	0.3997	2.3388	0.2027	-0.6116

IV – Implied parameters estimates

- Statistics about implied betas and idiosyncratic volatilities (two-factor):

Betas

Asset	*Mean*	*Stand. dev.(%)*	*Skewness*	*Excess kurtosis*
Ai	*0.5706*	*18.4062*	*1.1470*	*0.7643*
Bn	*0.5329*	*20.7275*	*0.8462*	*0.1712*
Or	*0.6621*	*19.2176*	*0.3263*	*-0.2072*
Rno	*0.6007*	*18.3587*	*0.6172*	*0.2519*
Su	*0.5742*	*19.7689*	*0.5503*	*-0.0164*
Gle	*0.5483*	*20.2703*	*0.8227*	*0.0230*
Tmm	*0.7469*	*23.3146*	*0.8789*	*1.0486*
Fp	*0.5489*	*20.9438*	*0.7608*	*-0.1397*
Fr	*0.5707*	*17.9346*	*1.1605*	*0.7391*
Ex	*0.5765*	*23.7883*	*0.7694*	*-0.1556*

Volatilities

Asset	*Mean*	*Stand. dev.(%)*	*Skewness*	*Excess kurtosis*
Ai	*0.2462*	*3.1497*	*-1.6912*	*2.9207*
Bn	*0.2360*	*4.5111*	*-1.5934*	*2.2661*
Or	*0.2319*	*4.3339*	*-0.0500*	*1.1928*
Rno	*0.3102*	*3.2547*	*0.2675*	*-1.4281*
Su	*0.3733*	*1.9486*	*-0.3100*	*0.0753*
Gle	*0.2851*	*2.2600*	*-1.0070*	*0.9803*
Tmm	*0.4734*	*3.0723*	*-1.2540*	*4.5287*
Fp	*0.1920*	*5.3648*	*-0.3120*	*-0.2902*
Fr	*0.2948*	*2.4797*	*-0.2280*	*-0.8646*
Ex	*0.3604*	*3.9593*	*-1.5757*	*2.8351*

IV – Two-factor *vs* BS(73) option pricing

- Compute the mean absolute pricing error for each option pricing methodology.
- ***Criterion***: minimize the mean absolute error.
- ***Results***: In 94.74% of cases (over 38 call prices), the mean absolute error is smaller for the two-factor pricing than for BS (73) one.
- BS (73) performs better only for two European call prices, which are near-the-money calls. Related stocks cross many times the corresponding strike thresholds…

V – Concluding remarks

- Any stock's price results from the combination of two independent systematic and idiosyncratic risk factors.

→ Mixture of Gaussian distributions.

- We deduce an analytic risk-neutral European call pricing.

→ BS (73) becomes a special case of our pricing framework when β_i is zero.

- Our two-factor pricing adds two more parameters as compared to BS (73) setting, namely σ_M and β_i .
- Empirical application on French stock & European call data

→ Daily closing prices.

- Implied time varying parameters: volatilities and betas.
- The two-factor option pricing performs generally better than BS (73) one.

The end...

Thank you for your attention!!!