A New Method for Approximating the Price of an American Option in a Path Integral Framework

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Introduction

$\Rightarrow$ **Mathematical Representation of Option Pricing**

- **Fourier-Hermite Series Expansion Method.**
- **Option Pricing using Interpolation Polynomials.**
- **Option Pricing using Quadrature Rules.**

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The Path Integral

$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k$$

$$(k = K, K-1, \dots, 2, 1)$$

where,

$$\mu(y) = \frac{1}{\sqrt{2\Delta t}}\left(y + \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)\Delta t\right)$$

σ - volatility, r - interest rate, T - time to expiry, $\Delta t = \frac{T}{K}$

The Pay Off Function (Call Option)

$$f^K(x_K) = \begin{cases} e^{\sigma x_K} - 1, & x_K > 0 \\ 0, & x_K \leq 0. \end{cases}$$

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$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t \pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

Fourier-Hermite Series Expansion.

⇒ **Authors: Chiarella, El-Hassan and Kucera.**

⇒ **Orthogonal Hermite Polynomials.**

$$f^j(x_j) = \sum_{m=0}^{\infty} \alpha_m H_m(x_j), \qquad (j = k, k-1)$$

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Approximating the Path Integral with Interpolation Polynomials.

$\Rightarrow$ **Approximate $f^k(x_k)$ with a series of interpolation polynomials**

$\Rightarrow$ **Convert the integral over a finite interval**

$\Rightarrow$ **Node Allocation**

$\Rightarrow$ **Results**

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$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

Approximate $f^k(x_k)$ with a series of interpolation polynomials

$$\Rightarrow p(x_{k,i}) = a_{0,i} + a_{1,i}x_{k,i} + a_{2,i}x_{k,i}^2 + \ldots + a_{n-1,i}x_{k,i}^{n-1} + a_{n,i}x_{k,i}^n$$

$\Rightarrow$ **Hermite Interpolation Polynomials of order 2**

$$f^k(x_i) = f(x_{k,i})H_1^*(x_{k,i}) + f^{'}(x_{k,i})H_2^*(x_{k,i}) +$$
$$f(x_{k,i+1})H_3^*(x_{k,i+1}) + f^{'}(x_{k,i+1})H_4^*(x_{k,i+1})$$

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$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

Convert the integral over a finite interval.

$$\Psi(x_{k-1}, a, b) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{a}^{b} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k$$

where,

$$a = \sqrt{2\Delta t}(\alpha + \mu(x_{k-1}))$$

$$b = \sqrt{2\Delta t}(\beta + \mu(x_{k-1}))$$

and,

$$f^{k-1}(x_{k-1}) = \Psi(x_{k-1}, -\infty, a) + \Psi(x_{k-1}, a, b) + \Psi(x_{k-1}, b, \infty)$$

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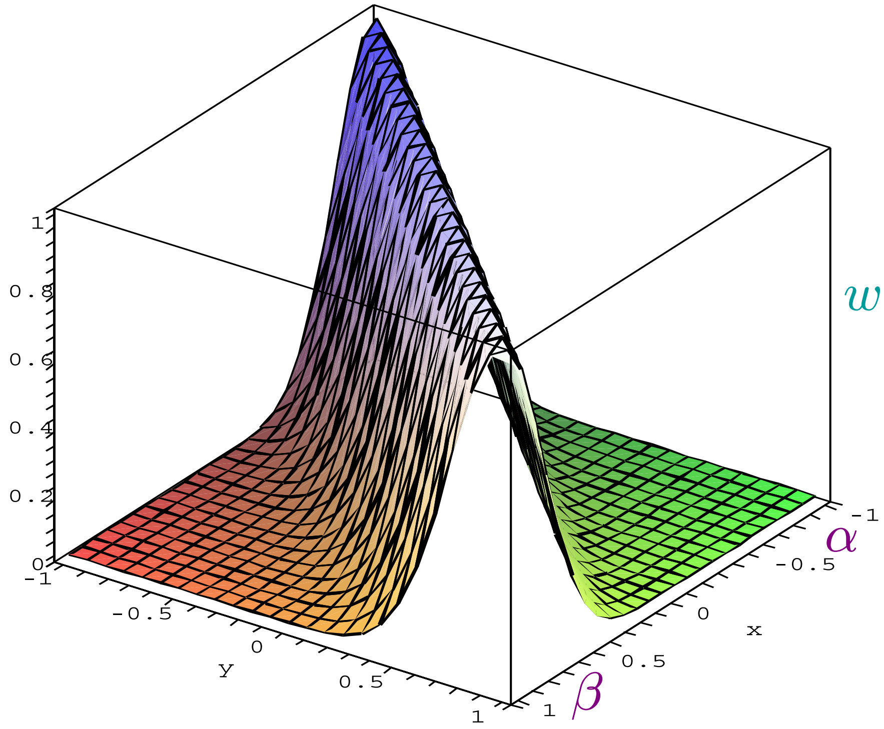
$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

Convert the integral over a finite interval.



$$w(x) = e^{-\left(\frac{x}{\sqrt{2\Delta t}} - \mu(y)\right)^2}$$

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$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

Convert the integral over a finite interval.

$\Rightarrow$ **Finding the optimal α and β**

$\Rightarrow$ **Use the upper bound**

$$f^k(x_k) \leq e^{\sigma x_k}$$

$\Rightarrow$ **Outer integrals have the analytical bounds**

$$\Psi(x_{k-1}, -\infty, a) < \frac{\omega}{2}(1 - \text{erf}(a + \sigma\sqrt{\frac{\Delta t}{2}}))$$

$$\Psi(x_{k-1}, b, \infty) < \frac{\omega}{2}(1 - \text{erf}(b - \sigma\sqrt{\frac{\Delta t}{2}}))$$

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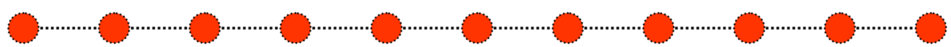
$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

Node Allocation.

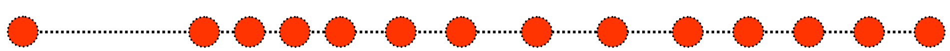
⇒ **Fixed number of node points**

⇒ **Equally spaced node points**



⇒ **Adaptive/Dynamic node allocation, dependent on an L_1 error**

$$\epsilon = \int_{x_{k,i}}^{x_{k,i+1}} \left| f(x_k) - f^*(x_k) \right| dx_k$$



⇒ **Dynamic node allocation at the first time step**

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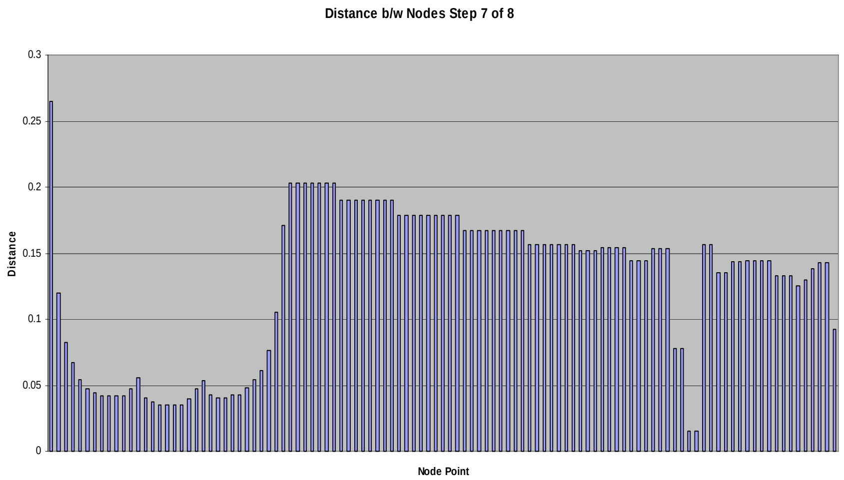

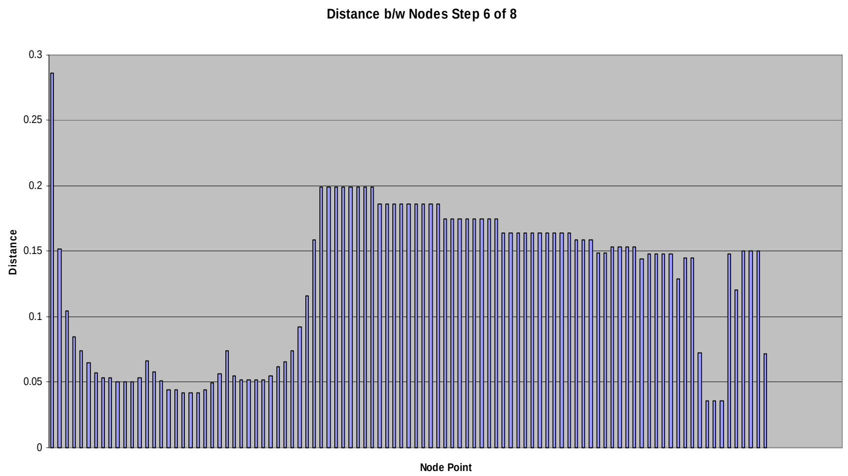

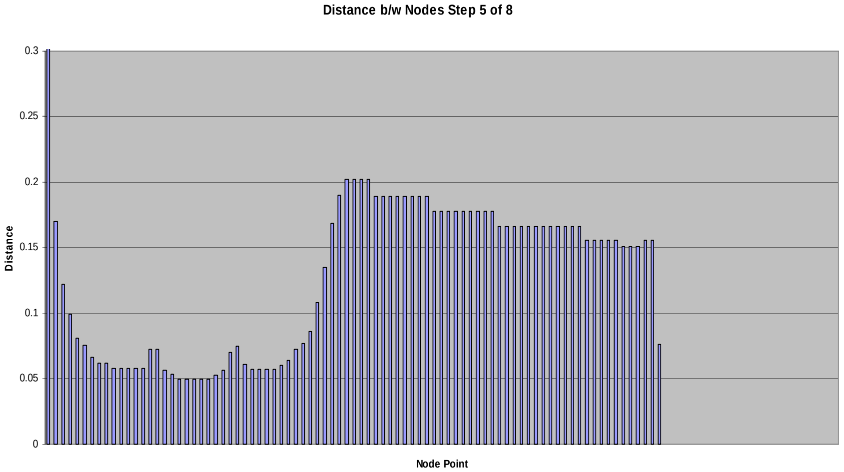

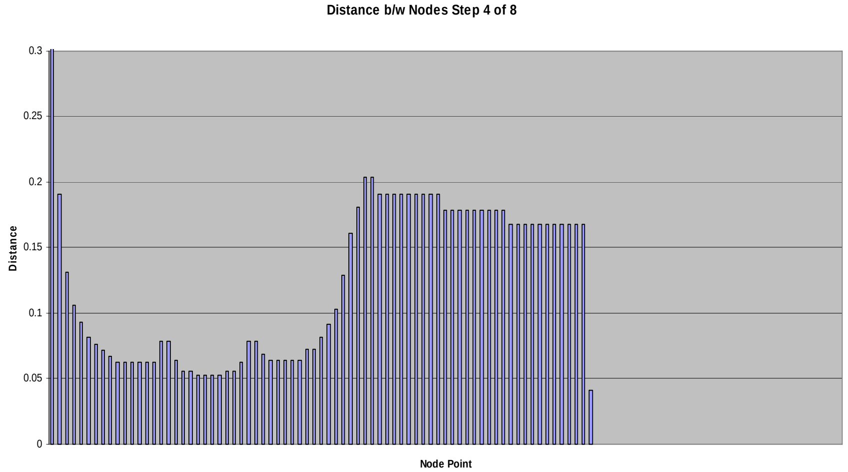

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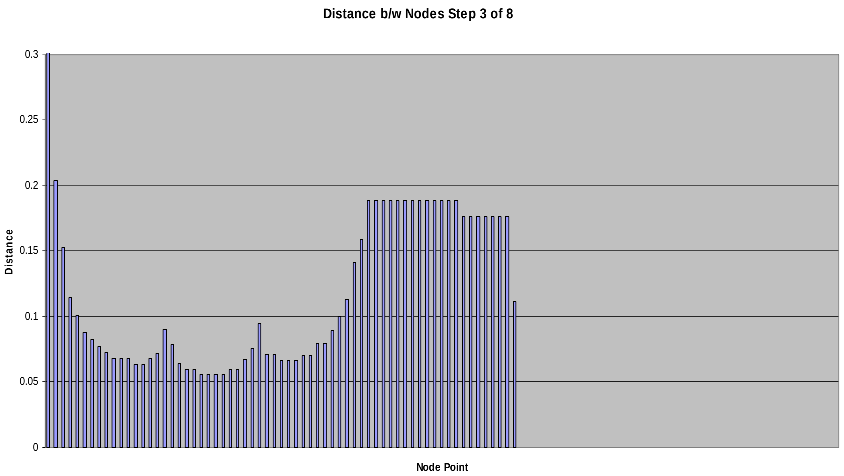
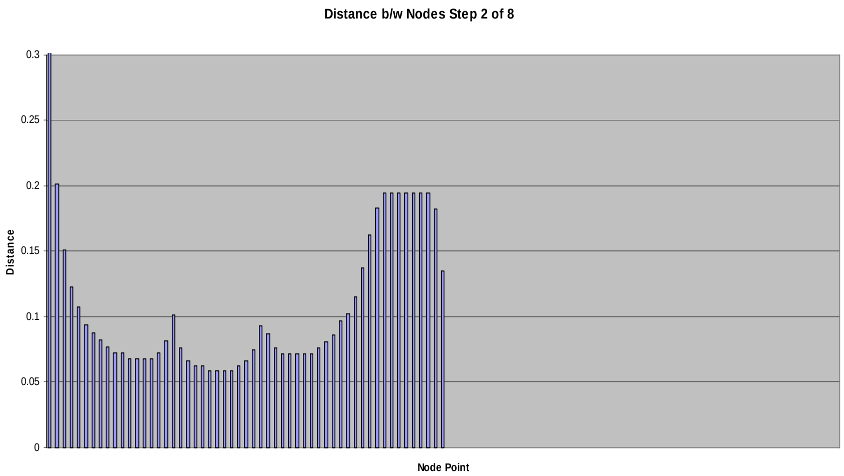
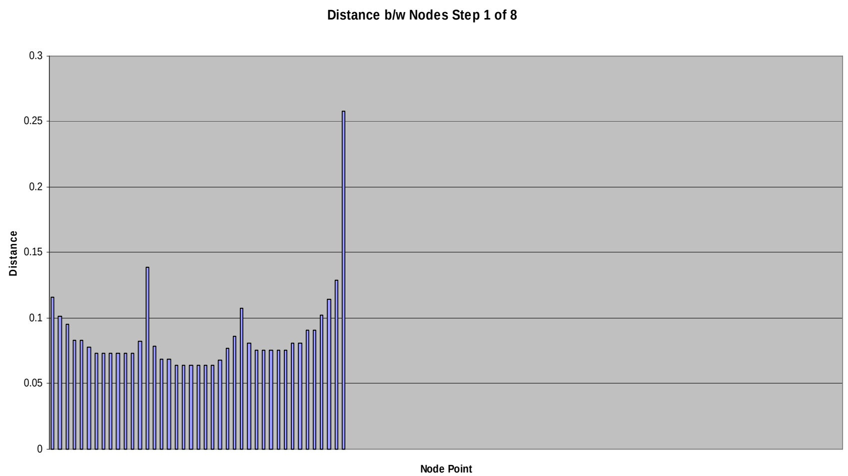






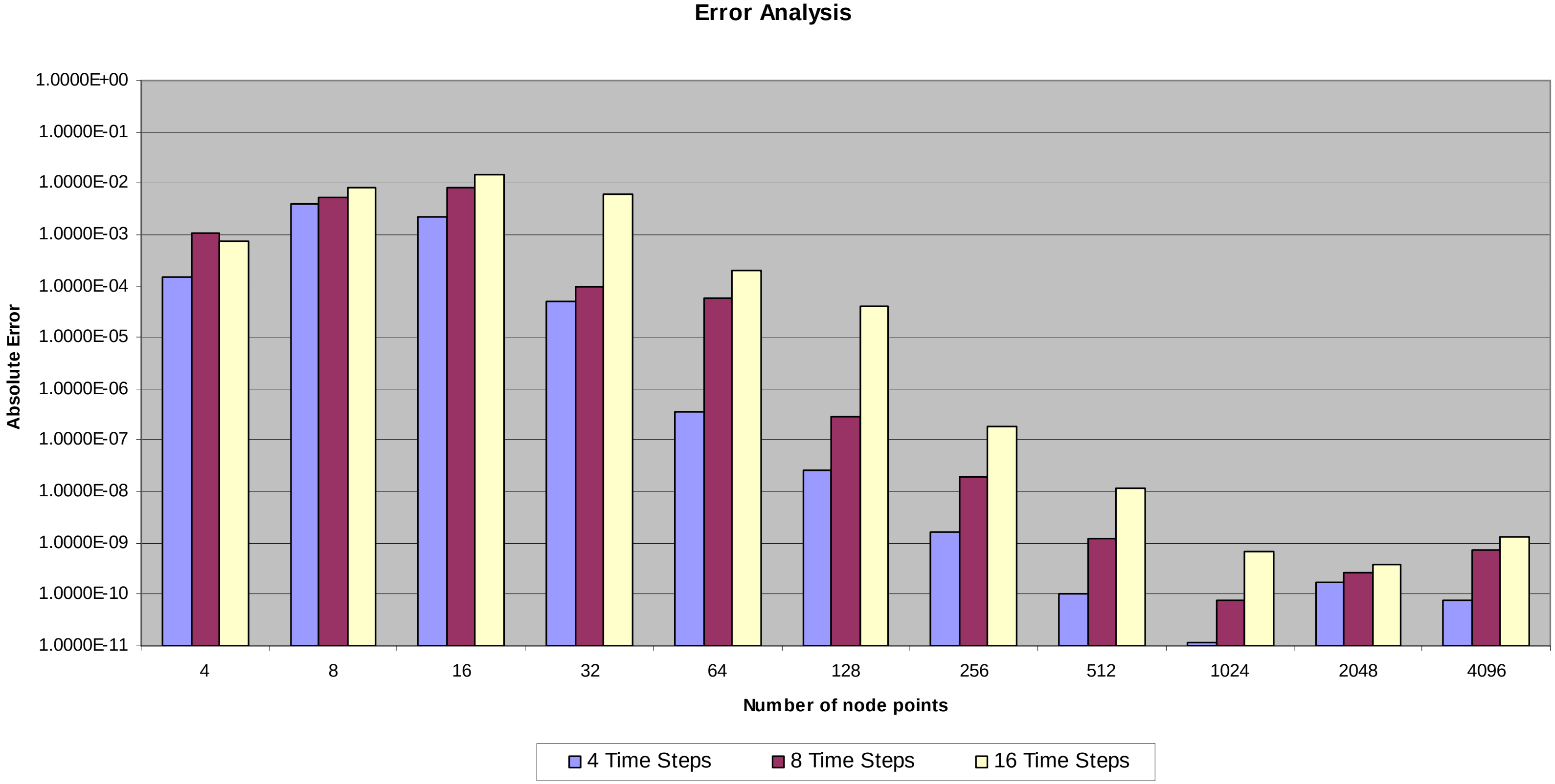
A New Method for Approximating the Price of an American Option in a Path Integral Framework

Results – European Call Option

Node Allocation – Fixed number of nodes .

$$\sigma = 0.2 \quad r = 0.08 \quad T = 0.25 \quad AP = 110 \quad EP = 100$$

$$WE = 0.5 \times 10^{-32}$$



A New Method for Approximating the Price of an American Option in a Path Integral Framework

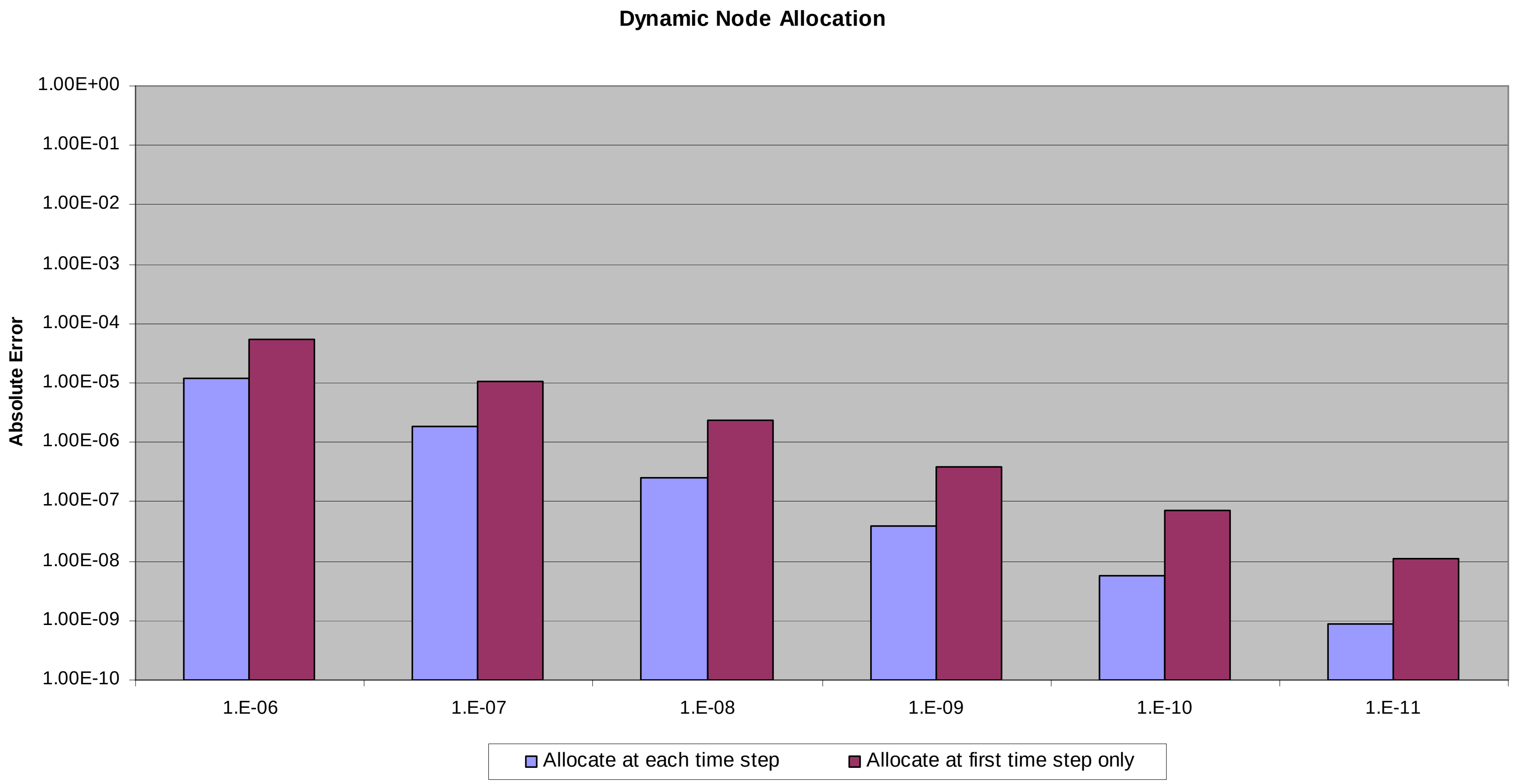
Results – American Put Option
Node Allocation – Fixed number of nodes .

$$\sigma = 0.2 \quad r = 0.08 \quad T = 0.25 \quad AP = \$100 \quad EP = \$100$$
$$K = 11 \quad WE = 0.5 \times 10^{-32} \quad \text{Binomial Result} = 3.224899$$

N	Interpolated Result	Absolute Relative Error
300	3.228478980483183	$3.469105373379093 \times 10^{-5}$
310	3.226950373988449	$1.988254799194712 \times 10^{-5}$
320	3.225562916660372	$6.441436836031820 \times 10^{-6}$
330	3.224299756321634	$5.795536638514722 \times 10^{-6}$
340	3.223146512532202	$1.696768417051101 \times 10^{-5}$
350	3.222090742346695	$2.719554785420811 \times 10^{-5}$

A New Method for Approximating the Price of an American Option in a Path Integral Framework

Results – European Call Option
Node Allocation – Adaptive/Dynamic.



A New Method for Approximating the Price of an American Option in a Path Integral Framework

Results – American Put Option
Node Allocation – Adaptive/Dynamic

$$\sigma = 0.2 \quad r = 0.08 \quad T = 0.25 \quad AP = \$100 \quad EP = \$100$$
$$K = 4 \quad WE = 0.5 \times 10^{-32} \quad \text{Binomial Result} = 3.224899$$

ϵ	Splines Used	Interpolated Result	Absolute Relative Error
5.0×10^{-5}	71	3.216960190154036	$7.689820963507687 \times 10^{-5}$
6.0×10^{-5}	66	3.221672986906726	$3.124258929539890 \times 10^{-5}$
6.1×10^{-5}	66	3.221697692638504	$3.100325041247033 \times 10^{-5}$
6.2×10^{-5}	65	3.222634158104079	$2.193116137465792 \times 10^{-5}$
6.3×10^{-5}	65	3.224505521814414	$3.802165884300708 \times 10^{-6}$
6.4×10^{-5}	64	3.225582043971720	$6.626734295442135 \times 10^{-6}$
6.5×10^{-5}	64	3.226771451090197	$1.814921716074054 \times 10^{-5}$
6.6×10^{-5}	64	3.227029739317819	$2.065140638665611 \times 10^{-5}$
7.0×10^{-5}	63	3.228738938503151	$3.720941921542011 \times 10^{-5}$

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Option Pricing using Quadrature Rules.

$\Rightarrow$ **Theoretical results using Peano Kernel Theory**

$$\int_a^b w(x) f(x)\, dx = \left[\int_a^{\frac{a+b}{2}} w(x)\, dx\right] f(a) + \left[\int_{\frac{a+b}{2}}^b w(x)\, dx\right] f(b) + E$$

where

$$|E| \leq ||f||_\infty \left[\int_a^{\frac{a+b}{2}} (t-a) w(t)\, dt + \int_{\frac{a+b}{2}}^b (b-t) w(t)\, dt\right]$$

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$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

Option Pricing using Quadrature Rules.

$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{a}^{b} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

$$\approx \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \left[\left(\int_{a}^{\frac{a+b}{2}} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} dx \right) f^k(a) + \right.$$

$$\left. \left(\int_{\frac{a+b}{2}}^{b} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} dx \right) f^k(b) \right]$$

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$$f^{k-1}(x_{k-1}) = \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_k}{\sqrt{2\Delta t}} - \mu(x_{k-1})\right)^2} f^k(x_k)\, dx_k.$$

Option Pricing using Quadrature Rules.

$$f^{k-1}(x_{k-1}) \approx \frac{e^{-r\Delta t}}{\sqrt{2\Delta t\pi}} \left[f^k(x_0) \int_{x_0}^{\frac{x_0+x_1}{2}} w(x_k)\, dx_k \right.$$

$$+ \sum_{i=1}^{N-1} f^k(x_i) \int_{\frac{x_{i-1}+x_i}{2}}^{\frac{x_i+x_{i+1}}{2}} w(x_k)\, dx_k$$

$$\left. + f^k(x_N) \int_{\frac{x_{N-1}+x_N}{2}}^{x_N} w(x_k)\, dx_k \right]$$

A New Method for Approximating the Price of an American Option in a Path Integral Framework

Results – American Put Option
Node Allocation – Fixed number of nodes.
Weighted Quadrature Rule.

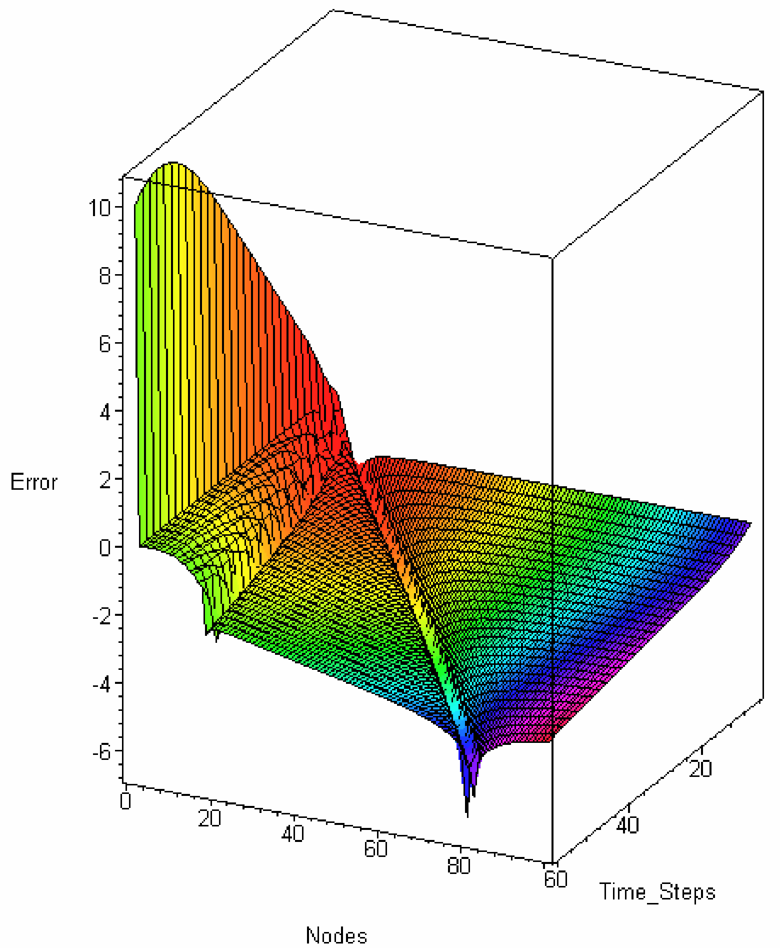
$\sigma = 0.2 \quad r = 0.08 \quad T = 0.25 \quad AP = \$100 \quad EP = \$100$

$K = 11 \quad WE = 0.5 \times 10^{-32} \quad \text{Binomial Result} = 3.224899$

N	Interpolated Result	Absolute Error
221	3.226038790132162	$1.140790132161396 \times 10^{-3}$
222	3.225844865667420	$9.468656674194165 \times 10^{-4}$
223	3.225653535798873	$7.553357988731082 \times 10^{-4}$
224	3.225464754503971	$5.667545039713673 \times 10^{-4}$
225	3.225278476774557	$3.804767745569926 \times 10^{-4}$
226	3.225094658590196	$1.966585901960194 \times 10^{-4}$
227	3.224913256892307	$1.525689230685856 \times 10^{-5}$
228	3.224734229559131	$1.637704408685714 \times 10^{-4}$
229	3.224557535381396	$3.404646186044857 \times 10^{-4}$

A New Method for Approximating the Price of an American Option in a Path Integral Framework

Results – American Put Option Weighted Quadrature Rule.



Where is the Optimal K and N?