# Pricing of credit derivatives in the Lévy Libor model

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# Basic interest rates

$B(t,T)$: price at time $t \in [0,T]$ of a default-free zero coupon bond with maturity $T \in [0,T^*]$ $\qquad (B(T,T)=1)$

$f(t,T)$: instantaneous forward rate

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

$L(t,T)$: default-free forward Libor rate for the interval $T$ to $T+\delta$ as of time $t \leq T$ $\qquad$ ($\delta$-forward Libor rate)

$$L(t,T) := \tfrac{1}{\delta}\left(\tfrac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

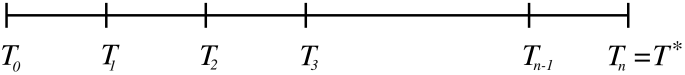
$F_B(t,T,U)$: forward price process for the two maturities $T$ and $U$

$$F_B(t,T,U) := \tfrac{B(t,T)}{B(t,U)}$$

$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$

# Dynamics of the Libor rate

Tenors: $T_0 < T_1 < \cdots < T_n = T^*$, $\delta_k := T_{k+1} - T_k$ (accrual periods)



Dynamics in the classical Gaussian case

$$\mathrm{d}L(t,T_k) = L(t,T_k)\lambda(t,T_k)\,\mathrm{d}W_t$$

solution

$$\begin{aligned} L(t,T_k) &= L(0,T_k)\mathcal{E}_t\left(\int_0^{\cdot} \lambda(s,T_k)\,\mathrm{d}W_s\right) \\ &= L(0,T_k)\exp\left(\int_0^t \lambda(s,T_k)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \lambda(s,T_k)^2\,\mathrm{d}s\right) \end{aligned}$$

# Forward measures

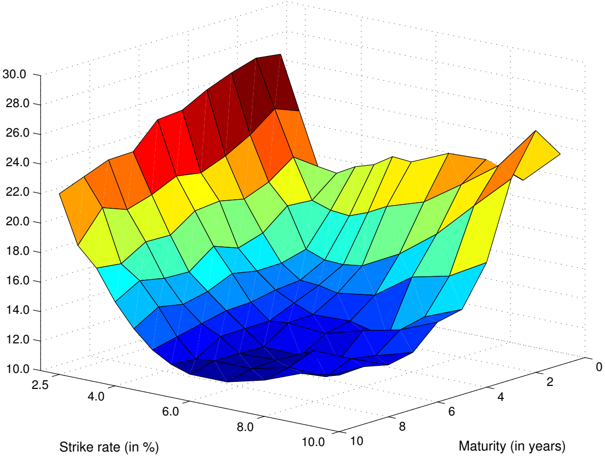
Forward measures associated with the dates $T_k$:

Density $\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}^*} = \frac{1}{B_{T_k} B(0,T_k)}$ or $\mathbb{E}_{\mathbb{P}^*}\left[\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}^*} \mid \mathcal{F}_t\right] = \frac{B(t,T_k)}{B_t B(0,T_k)}$

$$\left.\frac{d\mathbb{P}_{T_k}}{d\mathbb{P}_{T_{k+1}}}\right|_{\mathcal{F}_t} = \frac{B(t,T_k)}{B(t,T_{k+1})}\frac{B(0,T_{k+1})}{B(0,T_k)} = \frac{1+\delta_k L(t,T_k)}{1+\delta_k L(0,T_k)}$$

$$= \frac{F_B(t,T_k,T_{k+1})}{F_B(0,T_k,T_{k+1})}$$

# Caplet implied volatility surface
## February 19, 2002



# The driving process

$L = (L^1, \ldots, L^d)$ is a $d$-dimensional time-inhomogeneous Lévy process, i.e. $L$ has independent increments and the law of $L_t$ is given by the characteristic function

$$\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(iu)\,\mathrm{d}s \qquad \text{with}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x),$$

where $b_t \in \mathbb{R}^d$, $c_t$ is a symmetric nonnegative-definite $d \times d$-matrix and $F_t$ is a Lévy measure.

Integrability: $\sup_{0\le s\le T^*} \left(|b_s| + ||c_s|| + \int_{\{|x|\le 1\}} |x|^2 F_s(\mathrm{d}x)\right) < \infty$

$$\sup_{0\le s\le T^*} \int_{\{|x|>1\}} \exp\langle u, x\rangle F_s(\mathrm{d}x) < \infty \quad (u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d)$$

# Description in terms of modern stochastic analysis

$L = (L_t)$ is a special semimartingale with canonical representation

$$L_t = \int_0^t b_s \,\mathrm{d}s + \int_0^t c_s^{1/2} \,\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x)$$

and characteristics

$$A_t = \int_0^t b_s \,\mathrm{d}s, \qquad C_t = \int_0^t c_s \,\mathrm{d}s, \qquad \nu(ds, dx) = F_s(\mathrm{d}x)\,\mathrm{d}s$$

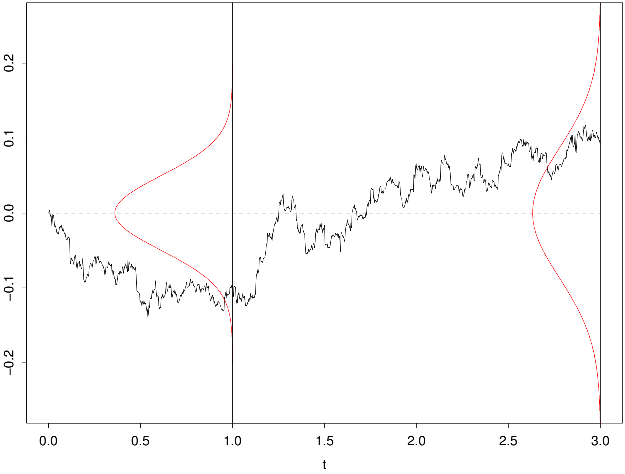
$W = (W_t)$ is a standard $d$-dimensional Brownian motion,

$\mu^L$ the random measure of jumps of $L$, and $\nu$ is the compensator of $\mu^L$.

$L$ is also called a process with *independent increments* and *absolutely continuous characteristics (PIAC)*.

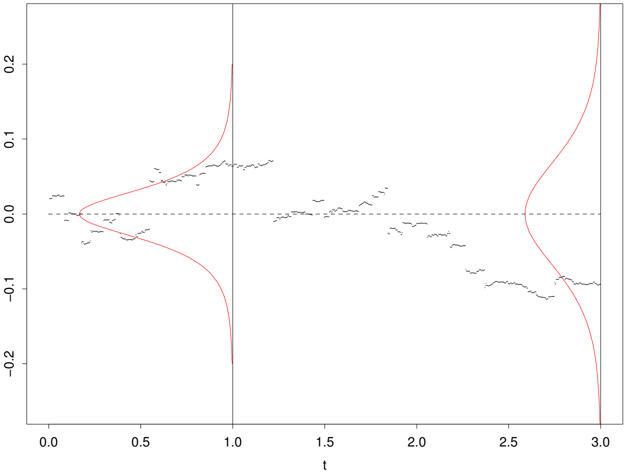
# Simulation of a Brownian motion

## $mean = 0, sd = 0.1$



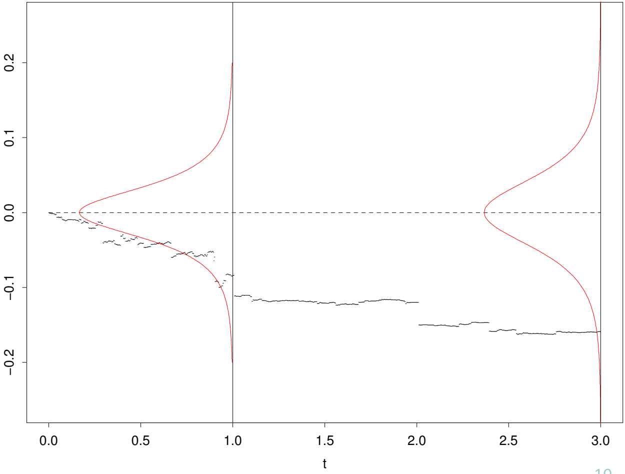
# Simulation of a Lévy process

## NIG(10,0,0.1,0)



# Simulation of a Lévy process

NIG(10,0,0.100,0) on [0,1]
NIG(10,0,0.025,0) on [1,3]



# The Lévy Libor model

Tenor structure: $T_0 < T_1 < \cdots < T_n = T^*$

**Assumptions**:

(LR.1): For any maturity $T_k$ there is a bounded deterministic function $\lambda(\cdot, T_k) : [0, T^*] \to \mathbb{R}^d$, which represents the volatility of the forward Libor rate process $L(\cdot, T_k)$. In addition

$$\sum_{k=1}^{n-1} |\lambda^j(s, T_k)| \leq M \qquad (s \in [0, T^*], 1 \leq j \leq d)$$

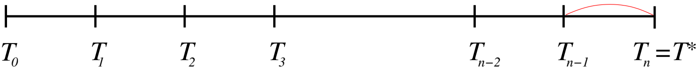
(LR.2): The initial term structure $B(0, T_k)$ $(1 \leq k \leq n)$ is strictly positive and strictly decreasing (in $k$).

Consequently the initial term structure of forward Libor rates is given by

$$L(0,\ T_k) = \frac{1}{\delta_k} \left( \frac{B(0, T_k)}{B(0, T_{k+1})} - 1 \right).$$

# Backward Induction

Given a stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}, (\widetilde{\mathcal{F}}_t)_{0\le t\le T^*})$



We postulate that under $\mathbb{P}_{T^*}$

$$L(t,T_{n-1}) = L(0,T_{n-1})\exp\left(\int_0^t b^L(s,T_{n-1},T_n)\,\mathrm{d}s + \int_0^t \lambda(s,T_{n-1})\,\mathrm{d}L_s^{T^*}\right)$$

where $$L_t^{T^*} = \int_0^t c_s^{1/2}\,\mathrm{d}W_s^{T^*} + \int_0^t\int_{\mathbb{R}^d} x(\mu^L - \nu^{T^*})(\mathrm{d}s,\mathrm{d}x)$$

is a non-homogeneous Lévy process. Here $W^{T^*}$ is a standard Brownian motion, $\nu^{T^*}(\mathrm{d}s,\mathrm{d}x) = F_s^{T^*}(\mathrm{d}x)\,\mathrm{d}s$ is the $\mathbb{P}_{T^*}$-compensator of the random measure of jumps $\mu^L$ and $F_s^{T^*}$ satisfies the integrability conditions from above.

# Backward Induction (2)

In order to make $L(t,T_{n-1})$ a $\mathbb{P}_{T^*}$-martingale, specify the drift term $b^L(s,T_{n-1},T_n)$ such that

$$\begin{aligned} b^L(s,T_{n-1},T_n) &= -\frac{1}{2}\langle\lambda(s,T_{n-1}), c_s\lambda(s,T_{n-1})\rangle \\ &\quad -\int_{\mathbb{R}^d}\left(e^{\langle\lambda(s,T_{n-1}),x\rangle}-1-\langle\lambda(s,T_{n-1}),x\rangle\right)F_s^{T^*}(\mathrm{d}x). \end{aligned}$$

Transform $L(t,T_{n-1})$ into a stochastic exponential

$$L(t,T_{n-1}) = L(0,T_{n-1})\mathcal{E}(H(t,T_{n-1}))$$

where

$$\begin{aligned} H(t,T_{n-1}) &= \int_0^t c_s^{1/2}\lambda(s,T_{n-1})\,\mathrm{d}W_s^{T^*} \\ &\quad + \int_0^t\int_{\mathbb{R}^d}\left(e^{\langle\lambda(s,T_{n-1}),x\rangle}-1\right)(\mu^L-\nu^{T^*})(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

# Backward Induction (3)

Equivalently

$$\begin{aligned} \mathrm{d}L(t,T_{n-1}) &= L(t-,T_{n-1})\Big(c_t^{1/2}\lambda(t,T_{n-1})\,\mathrm{d}W_t^{T^*} \\ &\quad + \int_{\mathbb{R}^d}\left(e^{\langle\lambda(t,T_{n-1}),x\rangle}-1\right)(\mu^L-\nu^{T^*})(\mathrm{d}t,\mathrm{d}x)\Big) \end{aligned}$$

with initial condition

$$L(0,T_{n-1}) = \frac{1}{\delta_{n-1}}\left(\frac{B(0,T_{n-1})}{B(0,T^*)}-1\right)$$

Recall $F_B(t,T_{n-1},T^*) = 1+\delta L(t,T_{n-1})$

# Backward Induction (4)

$$\begin{aligned}
dF_B(t,T_{n-1},T^*) &= \delta\, dL(t,T_{n-1}) \\
&= F_B(t-,T_{n-1},T^*)\Bigg(c_t^{1/2}\underbrace{\frac{\delta_{n-1}L(t-,T_{n-1})}{1+\delta_{n-1}L(t-,T_{n-1})}\lambda(t,T_{n-1})}_{=\alpha(t,T_{n-1},T^*)}dW_t^{T^*} \\
&\quad + \int_{\mathbb{R}^d}\underbrace{\frac{\delta_{n-1}L(t-,T_{n-1})}{1+\delta_{n-1}L(t-,T_{n-1})}\left(e^{\langle\lambda(t,T_{n-1}),x\rangle}-1\right)}_{=\beta(t,x,T_{n-1},T^*)-1}(\mu^L-\nu^{T^*})(dt,dx)\Bigg)
\end{aligned}$$

Define the forward martingale measure associated with $T_{n-1}$ by

$$\frac{d\mathbb{P}_{T_{n-1}}}{d\mathbb{P}_{T^*}} = \mathcal{E}_{T_{n-1}}(M^1) \qquad \text{where}$$

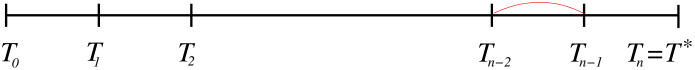
$$\begin{aligned}
M_t^1 &= \int_0^t c_s^{1/2}\alpha(s,T_{n-1},T^*)\, dW_s^{T^*} \\
&\quad + \int_0^t\int_{\mathbb{R}^d}(\beta(s,x,T_{n-1},T^*)-1)\,(\mu^L-\nu^{T^*})(ds,dx).
\end{aligned}$$

# Backward Induction (5)

Then $W_t^{T_{n-1}} = W_t^{T^*} - \int_0^t c_s^{1/2} \alpha(s, T_{n-1}, T^*)\, \mathrm{d}s$ is the forward Brownian motion for the date $T_{n-1}$ and the $\mathbb{P}_{T_{n-1}}$-compensator of $\mu^L$ is given by

$$\nu^{T_{n-1}}(\mathrm{d}t, \mathrm{d}x) = \beta(t, x, T_{n-1}, T^*)\nu^{T^*}(\mathrm{d}t, \mathrm{d}x).$$

**Second step**



We postulate that under $\mathbb{P}_{T_{n-1}}$

$$L(t, T_{n-2}) = L(0, T_{n-2}) \exp\left( \int_0^t b^L(s, T_{n-2}, T_{n-1})\, \mathrm{d}s + \int_0^t \lambda(s, T_{n-2})\, \mathrm{d}L_s^{T_{n-1}} \right)$$

$$\text{with } L_t^{T_{n-1}} = \int_0^t c_s^{1/2}\, \mathrm{d}W_s^{T_{n-1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu^{T_{n-1}})(\mathrm{d}s, \mathrm{d}x).$$

# Backward Induction (6)

$L(t,T_{n-2})$ is a $\mathbb{P}_{T_{n-1}}$-martingale if $b^L(s,T_{n-2},T_{n-1})$ is chosen appropriately.

Second measure change: $\frac{d\mathbb{P}_{T_{n-2}}}{d\mathbb{P}_{T_{n-1}}} = \mathcal{E}_{T_{n-2}}(M^2)$

where

$$\begin{aligned} M_t^2 &= \int_0^t c_s^{1/2}\alpha(s,T_{n-2},T_{n-1})\,dW_s^{T_{n-1}} \\ &\quad + \int_0^t \int_{\mathbb{R}^d} (\beta(s,x,T_{n-2},T_{n-1}) - 1)(\mu^L - \nu^{T_{n-1}})(ds,dx) \end{aligned}$$

This way we get for each maturity $T_k$ in the tenor structure a Libor rate process which is under the forward martingale measure $\mathbb{P}_{T_{k+1}}$ of the form

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t b^L(s,T_k,T_{k+1})\,ds + \int_0^t \lambda(s,T_k)\,dL_s^{T_{k+1}}\right).$$

# The Lévy Libor model with default risk

$B^0(t,T_k)$: time-$t$ price of a defaultable zero coupon bond with zero recovery and maturity $T_k$

$\tau$: time of default

$\overline{B}(t,T_k)$: pre-default value of the defaultable bond

$$\Longrightarrow \quad B^0(t,T_k) = \mathbb{1}_{\{\tau>t\}}\overline{B}(t,T_k), \quad \overline{B}(T_k,T_k) = 1 \quad (k = 1,\ldots,n)$$

Terminal value of the defaultable bond

$$B^0(T_k,T_k) = \mathbb{1}_{\{\tau>T_k\}}\overline{B}(T_k,T_k) = \mathbb{1}_{\{\tau>T_k\}}$$

# The Lévy Libor model with default risk (2)

- The *defaultable forward Libor rates* for the interval $[T_k, T_{k+1}]$ are given by

$$\overline{L}(t,T_k) := \frac{1}{\delta_k}\left(\frac{\overline{B}(t,T_k)}{\overline{B}(t,T_{k+1})} - 1\right).$$

- The *forward Libor spreads* are given by

$$S(t,T_k) := \overline{L}(t,T_k) - L(t,T_k).$$

- The *default risk factors* or *forward survival processes* are given by

$$D(t,T_k) := \frac{\overline{B}(t,T_k)}{B(t,T_k)}.$$

- The discrete-tenor *forward default intensities* are given by

$$H(t,T_k) := \frac{1}{\delta_k}\left(\frac{D(t,T_k)}{D(t,T_{k+1})} - 1\right) = \frac{S(t,T_k)}{1+\delta L(t,T_k)}.$$

# Canonical construction of the time of default

Let $\Gamma = (\Gamma_t)_{t\geq 0}$ be an $(\widetilde{\mathcal{F}}_t)$-adapted, right-continuous, increasing process on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{P}_{T^*})$, $\Gamma_0 = 0$, $\lim_{t\to\infty} \Gamma_t = \infty$.

Let $\eta$ be a random variable on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ uniformly distributed on $[0,1]$.

Define $\Omega := \widetilde{\Omega} \times \widehat{\Omega}, \qquad \mathcal{G} := \widetilde{\mathcal{F}} \otimes \widehat{\mathcal{F}}, \qquad \mathbb{Q}_{T^*} := \mathbb{P}_{T^*} \otimes \widehat{\mathbb{P}}$

$(\mathcal{F}_t)$ trivial extension of $(\widetilde{\mathcal{F}}_t)$ to $(\Omega, \mathcal{G}, \mathbb{Q}_{T^*})$

$$\tau := \inf\{t \in \mathbb{R}_+ : e^{-\Gamma_t} \leq \eta\}$$

Denote $\mathcal{H}_t := \sigma\left(\mathbb{1}_{\{\tau\leq u\}} | \, 0 \leq u \leq t\right), \quad \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$

$$\Longrightarrow \quad \tau \text{ is a } (\mathcal{G}_t)\text{-stopping time}$$

$$\mathbb{Q}_{T^*}\{\tau > s | \mathcal{F}_{T^*}\} = \mathbb{Q}_{T^*}\{\tau > s | \mathcal{F}_s\} = e^{-\Gamma_s} \qquad (0 \leq s \leq T^*)$$

$\Longrightarrow (\Gamma_t)$ is the $(\mathcal{F}_t)$-hazard process of $\tau$ under $\mathbb{Q}_{T^*}$ (and also under all $\mathbb{Q}_{T_k}$)

# Consequences for the price of a defaultable bond

Payoff at maturity: $B^0(T_k, T_k) = \mathbb{1}_{\{\tau > T_k\}}$

$$\begin{aligned} \Longrightarrow \quad B^0(t, T_k) &= B(t, T_k) \mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbb{1}_{\{\tau > T_k\}} | \mathcal{G}_t] \\ &= B(t, T_k) \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbb{1}_{\{\tau > T_k\}} | \mathcal{F}_t]}{e^{-\Gamma_t}} \end{aligned}$$

Therefore, define

$$\overline{B}(t, T_k) := B(t, T_k) \frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbb{1}_{\{\tau > T_k\}} | \mathcal{F}_t]}{e^{-\Gamma_t}}$$

$$\Longrightarrow \quad H(t, T_k) = \frac{1}{\delta_k} \left( \frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[e^{-\Gamma_{T_k}} \,|\, \mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_{k+1}}}[e^{-\Gamma_{T_{k+1}}} \,|\, \mathcal{F}_t]} - 1 \right)$$

$(\Gamma_{T_k})_{k=1,\dots,n}$ can be chosen such that $H(t, T_k)$ has the form

$$\begin{aligned} H(t, T_k) \;=\; H(0, T_k) \exp\Bigg( &\int_0^t b^H(s, T_k, T_{k+1})\, ds + \int_0^t c_s^{1/2} \gamma(s, T_k)\, dW_s^{T_{k+1}} \\ &+ \int_0^t \int_{\mathbb{R}^d} \langle \gamma(s, T_k), x \rangle (\mu - \nu^{T_{k+1}})(ds, dx) \Bigg). \end{aligned}$$

# Defaultable forward measures

The *defaultable forward measure* (or *survival measure*) $\overline{\mathbb{Q}}_{T_i}$ for the settlement day $T_i$ is defined on $(\Omega, \mathcal{G}_{T_i})$ by

$$\frac{\mathrm{d}\overline{\mathbb{Q}}_{T_i}}{\mathrm{d}\mathbb{Q}_{T_i}} := \frac{B(0,T_i)}{B^0(0,T_i)} B^0(T_i,T_i) = \frac{B(0,T_i)}{\overline{B}(0,T_i)} 1\!\!1_{\{\tau > T_i\}}.$$

$$\Longrightarrow \quad \overline{\mathbb{Q}}_{T_i}(A) = \mathbb{Q}_{T_i}(A | \{\tau > T_i\}) \qquad (A \in \mathcal{G}_{T_i}),$$

forward measure conditioned on survival until $T_i$ $\longrightarrow$ survival measure

Denote $\mathbb{P}_{T_i} := \mathbb{Q}_{T_i}|_{\mathcal{F}_{T_i}}$
The *restricted defaultable forward measure* $\overline{\mathbb{P}}_{T_i}$ for the settlement day $T_i$ is defined on $(\Omega, \mathcal{F}_{T_i})$ by

$$\frac{\mathrm{d}\overline{\mathbb{P}}_{T_i}}{\mathrm{d}\mathbb{P}_{T_i}} = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \mathbb{Q}_{T_i}(\{\tau > T_i\} | \mathcal{F}_{T_i}) = \frac{B(0,T_i)}{\overline{B}(0,T_i)} \prod_{k=0}^{i-1} \frac{1}{1 + \delta_k H(T_k,T_k)}.$$

# Pricing contingent claims with defaultable forward measures

$X$ promised payoff at day $T_i$ with zero recovery upon default

$\pi_t^X$ its price at time $t \in [0, T_i]$

$$\pi_t^X = \mathbb{1}_{\{\tau > t\}} B(t,T_i) \mathbb{E}_{\mathbb{Q}_{T_i}}[X \mathbb{1}_{\{\tau > T_i\}} | \mathcal{G}_t] \qquad (t \in [0, T_i])$$

The defaultable forward measures $\overline{\mathbb{Q}}_{T_i}$ and $\overline{\mathbb{P}}_{T_i}$ are the appropriate tools.

If $X$ is $\mathcal{G}_{T_i}$-measurable

$$\pi_t^X = \mathbb{1}_{\{\tau > t\}} \overline{B}(t,T_i) \mathbb{E}_{\overline{\mathbb{Q}}_{T_i}}[X | \mathcal{G}_t] = B^0(t,T_i) \mathbb{E}_{\overline{\mathbb{Q}}_{T_i}}[X | \mathcal{G}_t].$$

If $X$ is $\mathcal{F}_{T_i}$-measurable

$$\pi_t^X = \mathbb{1}_{\{\tau > t\}} \overline{B}(t,T_i) \mathbb{E}_{\overline{\mathbb{P}}_{T_i}}[X | \mathcal{F}_t] = B^0(t,T_i) \mathbb{E}_{\overline{\mathbb{P}}_{T_i}}[X | \mathcal{F}_t].$$

# Recovery rules and bond prices

Defaultable zero coupon bonds
$\longrightarrow$ *fractional recovery of treasury value* scheme

At maturity of the bond

$$B^{\pi}(T,T) = \mathbb{1}_{\{\tau > T\}} + \pi \mathbb{1}_{\{\tau \le T\}} = \pi + (1-\pi)\mathbb{1}_{\{\tau > T\}}$$

Time-$t$ value ($t \in [0,T]$)

$$B^{\pi}(t,T) = \pi B(t,T) + (1-\pi)\mathbb{1}_{\{\tau > t\}}\overline{B}(t,T)$$

Defaultable coupon bonds $\longrightarrow$ *recovery of par* scheme

Recovery of par: If default occurs in the time interval $(T_k, T_{k+1}]$, recovery is given by the recovery rate $\pi$ times the sum of the notional and the accrued interest over $(T_k, T_{k+1}]$. It is paid at $T_{k+1}$.

Corresponding cashflow pattern

- at $T_{k+1}$ ($k = 0, \dots, m-1$): $c\mathbb{1}_{\{\tau > T_{k+1}\}} + \pi(1+c)\mathbb{1}_{\{T_k < \tau \le T_{k+1}\}}$
- at $T_m$: $\mathbb{1}_{\{\tau > T_m\}}$

# Pricing of defaultable coupon bonds

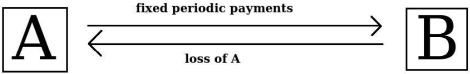
Fixed coupon of $c$ to be paid at dates $T_1, \ldots, T_m$

$$B^{\pi}_{\text{fixed}}(0) \;=\; \overline{B}(0,T_m) + \sum_{k=0}^{m-1} \overline{B}(0,T_{k+1}) \Big( c + \pi(1+c)\delta_k \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)] \Big) .$$

Floating coupon bond that pays Libor plus a constant spread $x$
Promised payoff at the date $T_{k+1}$: $\quad \delta_k(L(T_k,T_k)+x)$

$$\begin{aligned} B^{\pi}_{\text{floating}}(0) \;&=\; \overline{B}(0,T_m) + \sum_{k=0}^{m-1} \delta_k \overline{B}(0,T_{k+1}) \Big( x + \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[L(T_k,T_k)] \\ &\qquad + \pi(1+\delta_k x)\mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)] \\ &\qquad + \pi\delta_k \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}}[H(T_k,T_k)L(T_k,T_k)] \Big). \end{aligned}$$

# Credit default swaps (CDS)



Standard default swap: Default of a coupon bond

$A$ receives: $\quad 1-\pi(1+c) \quad$ (fixed coupon)

$\quad 1-\pi(1+\delta_k(L(T_k,T_k)+x)) \quad$ (floating coupon)

Time-0 value of the fee payments: $\quad s\sum_{k=1}^{m}\overline{B}(0,T_{k-1})$
$s$ default swap rate

$$s_{\text{fixed}} = \frac{1-\pi(1+c)}{\sum_{k=1}^{m}\overline{B}(0,T_{k-1})}\sum_{k=1}^{m}\left(\overline{B}(0,T_k)\delta_{k-1}\mathbb{E}_{\overline{\mathbb{P}}_{T_k}}\left[H(T_{k-1},T_{k-1})\right]\right)$$

$$\begin{aligned} s_{\text{floating}} &= \frac{1}{\sum_{k=1}^{m}\overline{B}(0,T_{k-1})}\sum_{k=1}^{m}\Bigg(\overline{B}(0,T_k)\delta_{k-1}\Big((1-\pi(1+\delta_{k-1}x)) \\ &\quad \times \mathbb{E}_{\overline{\mathbb{P}}_{T_k}}\left[H(T_{k-1},T_{k-1})\right]-\pi\delta_{k-1}\mathbb{E}_{\overline{\mathbb{P}}_{T_k}}\left[H(T_{k-1},T_{k-1})L(T_{k-1},T_{k-1})\right]\Big)\Bigg) \end{aligned}$$

# Credit default swaptions (1)

**Assumption:** The volatility structures factorize in the following way:

$$\lambda(s, T_i) = \lambda_i \sigma(s) \qquad \text{and} \qquad \gamma(s, T_i) = \gamma_i \sigma(s) \qquad (0 \leq s \leq T_i).$$

Payoff of a credit default swaption that is knocked out at default with strike $S$ and maturity $T_i$ on a CDS that terminates at $T_m$:

$$\mathbb{1}_{\{\tau > T_i\}} \left( (s(T_i; T_i, T_m) - S)^+ \sum_{k=i}^{m-1} \overline{B}(T_i, T_k) \right)$$

where $s(T_i; T_i, T_m)$ denotes the default swap rate at $T_i$.

Price at time 0:

$$\pi_0^{\mathrm{CDS}} = \overline{B}(0, T_i) \mathbb{E}_{\overline{\mathbb{P}}_{T_i}} \left[ \left( \frac{(1 - \pi(1 + c))\delta_{m-1} C^{i,m-1} H(T_i, T_{m-1})}{\prod_{l=i}^{m-1} (1 + \delta_l L(T_i, T_l))(1 + \delta_l H(T_i, T_l))} \right.\right.$$
$$\left.\left. + \sum_{k=i}^{m-2} \frac{(1 - \pi(1 + c))\delta_k C^{i,k} H(T_i, T_k) - S}{\prod_{l=i}^{k} (1 + \delta_l L(T_i, T_l))(1 + \delta_l H(T_i, T_l))} - S \right)^+ \right].$$

# Credit default swaptions (2)

Forward Libor rates and default intensities can be written as

$$\begin{aligned} L(T_i, T_l) &= L(0, T_l) \exp\left(\frac{\lambda_l}{\sigma_{\text{sum}}} X_{T_i} + B_l^L\right), \\ H(T_i, T_l) &= H(0, T_l) \exp\left(\frac{\gamma_l}{\sigma_{\text{sum}}} X_{T_i} + B_l^H\right) \end{aligned}$$

with $\sigma_{\text{sum}} := \sum_{l=i}^{m-1} (\lambda_l + \gamma_l)$, $X_{T_i} := \sigma_{\text{sum}} \int_0^{T_i} \sigma(s)\, \mathrm{d}L_s^{T^*}$ and constants $B_l^L$, $B_l^H$.

Assume the distribution of $X_{T_i}$ w.r.t. $\overline{\mathbb{P}}_{T_i}$ has a Lebesgue-density $\varphi$, then

$$\pi_0^{\text{CDS}} = \overline{B}(0, T_i) \int_{\mathbb{R}} g(-x)\varphi(x)\, \mathrm{d}x = \overline{B}(0, T_i)(g * \varphi)(0)$$

for some (explicitly given) function $g$.

Performing Laplace and inverse Laplace transformations and denoting by $\overline{M}_{T_i}^{X_{T_i}}$ the $\overline{\mathbb{P}}_{T_i}$-moment generating function of $X_{T_i}$ yields

$$\pi_0^{\text{CDS}} = \overline{B}(0, T_i) \frac{1}{\pi} \int_0^{\infty} \Re\Big( L[g](R + \mathrm{i}u) \overline{M}_{T_i}^{X_{T_i}}(-R - \mathrm{i}u) \Big)\, \mathrm{d}u.$$

# Further credit derivatives

- Total rate of return swaps
- Asset package swaps
- Options on defaultable bonds
- Credit spread options