# AN EQUILIBRIUM APPROACH TO GROUP DIVERSIFICATION

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# Overview

- Motivation: Solvency II
- Diversification under constrained capital mobility
- Link to equilibrium theory for convex risk measures
- Concrete example

# Solvency II

- New EU-system to assess overall solvency based on prospective risk-oriented approach (initiated 2001)
- 3 pillars: quantitative requirements, supervisory activities, public disclosure
- Pillar 1: Solvency Capital Requirement (SCR) and Minimum Capital Requirement (MCR), valuation of assets and liabilities, group ***diversification***, etc.
- Committee of European Insurance and Occupational Pension Supervisors (CEIOPS) in charge to advice the Solvency II project through three specific calls for advice (Aug 2004-Mar 2006)
- Formal adoption of new framework directive by EC in July 2007

http://europa.eu.int/comm/internal_market/insurance/solvency_en.htm

# Available Risk Capital (C)

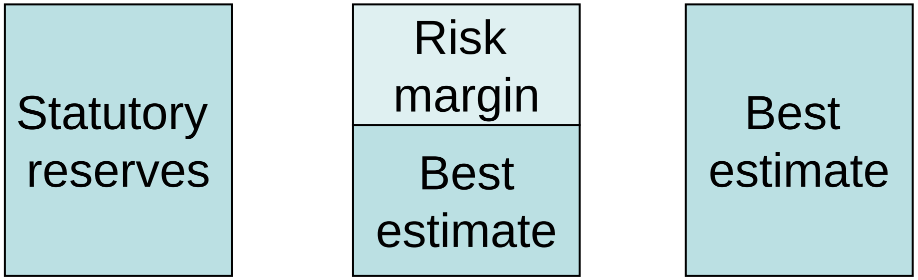
$$C = A - L$$

available capital — value of assets — value of liabilities

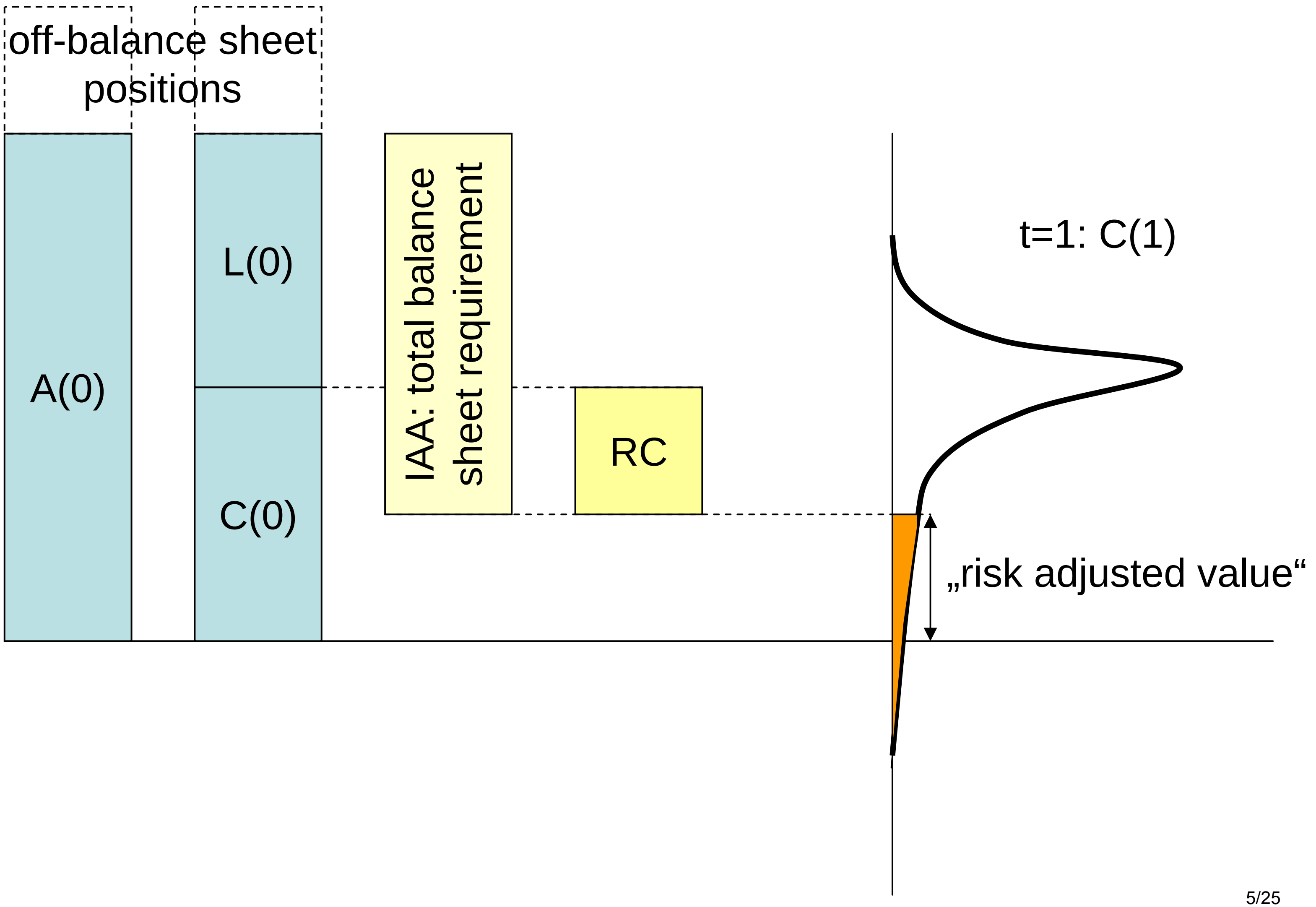
- Depends on choice of and valuation principles for assets and liabilities

- Market consistent valuation of assets:
  - marked to market if available
  - marked to model else (e.g. risk-neutral valuation)

- Value of Liabilities:



# Required Risk Capital (RC)



## Diversification

Diversification = spreading the portfolio over a variety of exposures (products, markets, legal entities, etc.), rather than only a few selected areas.

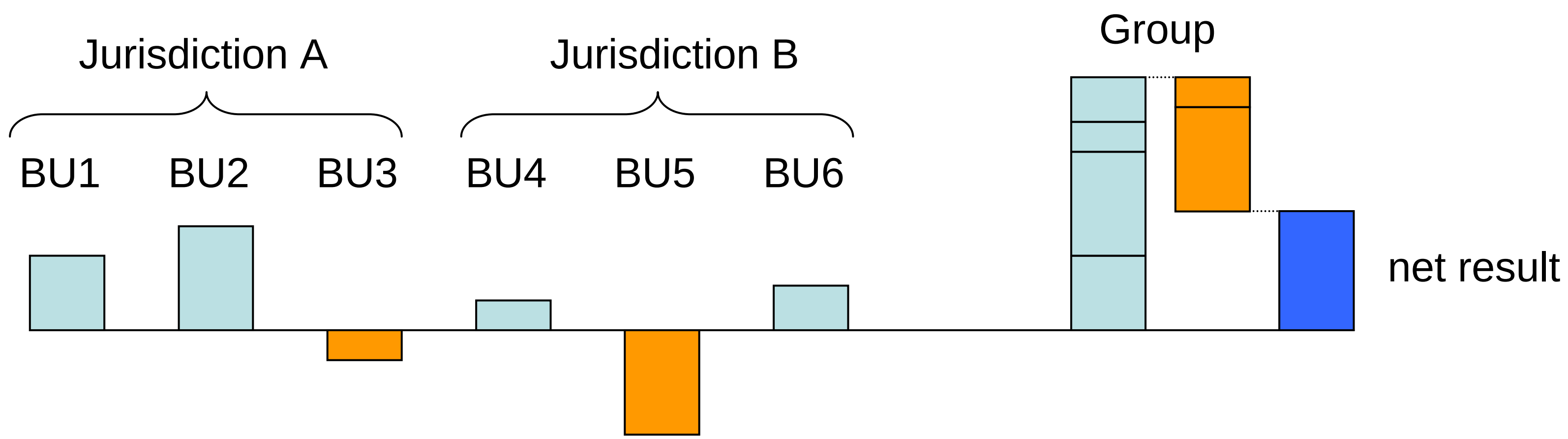
◊ The pooling of many independent individual risks results in a low coefficient of variation of the P&L.

◊ On a larger scale, independent risk types (such as market and technical insurance risks) have a statistically compensatory effect on the relative P&L variability.

◊ ALM, Hedging: compensation of opposite effects of risk factors by opposite portfolio sensitivities.

# Diversification

## Diversification requires fungibility of capital: BU=business unit




**Regulatory risk**: regulators may prevent capital to be transferred between jurisdictions

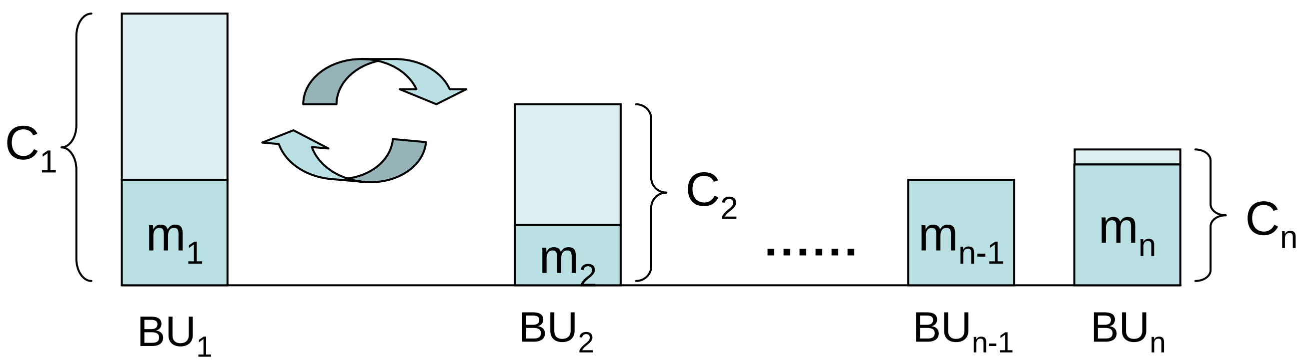
**Management risk**: companies' managements may refuse to provide necessary capital injections

**→ need of standardization for risk & capital transfers!**

# Standardized Diversification. A simple example

- *n* BUs with (random) future **available capital** $C_1,\ldots,C_n$
- Each BU faces a **minimum capital requirement** $m_1,\ldots,m_n$
- All BUs can share **excess gain** $(C_i-m_i)^+$ of BU *i* at appropriate price
- New positions of BUs become

$$C_i + \sum_j \lambda_{ij}\left(C_j - m_j\right)^+ + \gamma_i$$



## Optimization Problem

$$\inf_{\lambda_{ij},\gamma_i} \sum_{i=1}^{n} \rho\left(C_i + \sum_{j=1}^{n} \lambda_{ij}\left(C_j - m_j\right)^+ + \gamma_i\right)$$

subject to the **(sub-)clearing condition**

$$\sum_{i=1}^{n}\left(C_i + \sum_{j=1}^{n} \lambda_{ij}\left(C_j - m_j\right)^+ + \gamma_i\right) \le \sum_{i=1}^{n} C_i$$

Fact: "$<$" may happen for some $\omega \to$ dividends, group benefit

**Q:** Balance sheet ($C(0) = A(0) - L(0)$) after capital and risk transfer?

**Q:** Fair value for surplus participation $(C_j - m_j)^+$?

**Q:** Extended formalization of this optimization problem?

# Equilibrium Approach

$n$ agents with initial endowments $X_i \in L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and convex risk measures $\rho_i : L^\infty \to \mathbb{R}$. Write $X := \sum_i X_i$.

1. monotone: $\rho_i(Z) \leq \rho_i(Y)$ if $Z \geq Y$
2. convex: $\rho_i(\lambda Z + (1-\lambda)Y) \leq \lambda \rho_i(Z) + (1-\lambda)\rho_i(Y)$, $\lambda \in [0,1]$
3. cash invariant: $\rho_i(Z + \gamma) = \rho_i(Z) - \gamma$ for all $\gamma \in \mathbb{R}$
4. (coherent: $\rho_i(\lambda Y) = \lambda \rho_i(Y)$ for all $\lambda \geq 0$.)

Unconstrained optimization is trivial if $\rho_i \equiv \rho$ coherent:

$$\sum_{i=1}^{n} \rho\left(\lambda_i X + \gamma_i\right) = \rho(X) \leq \sum_{i=1}^{n} \rho(\xi_i) \qquad \forall \sum_{i=1}^{n} \xi_i \leq X \text{ (sub-clearing)}$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ (convex risk sharing), $\sum_i \gamma_i = 0$ (cash rebalancing) s.t. $\rho\left(\lambda_i X + \gamma_i\right) \leq \rho(X_i)$ (individual rationality)

# Capital and Risk Transfer Constraints

**Assumption**: Agent $i$ can only assume positions in $M_i \subset L^\infty$. $X_i \in M_i$.

$M_i$ closed, convex, cash invariant: $M_i + \mathbb{R} = M_i$ (cash is fungible)

The $(M_i)$-**constrained convolution** of $\rho_1, \ldots, \rho_n$ is defined by

$$\Box_i^{(M_i)} \rho_i(X) := \inf_{\sum_i \xi_i \le X,\, \xi_i \in M_i} \sum_{i=1}^n \rho_i(\xi_i).$$

**Remark:** the "sub-clearing condition" $\sum_i \xi_i \le X$ is essential!

**Assumption:** $\Box_i^{(M_i)} \rho_i(X) > -\infty$

**Proposition 1.** $\Box_i^{(M_i)} \rho_i : L^\infty \to \mathbb{R}$ *is a convex risk measure with "penalty fct"*

$$\alpha^{(M_i)}(\mu) = \sum_{i=1}^n \sup_{\xi \in M_i} \left( \langle \mu, -\xi \rangle - \rho_i(\xi) \right), \quad \mu \in (L^\infty)^*.$$

*Moreover,*

$$\emptyset \neq \partial \Box_i^{(M_i)} \rho_i(\xi) \subset (-\mathcal{P}), \quad \forall \xi \in L^\infty$$

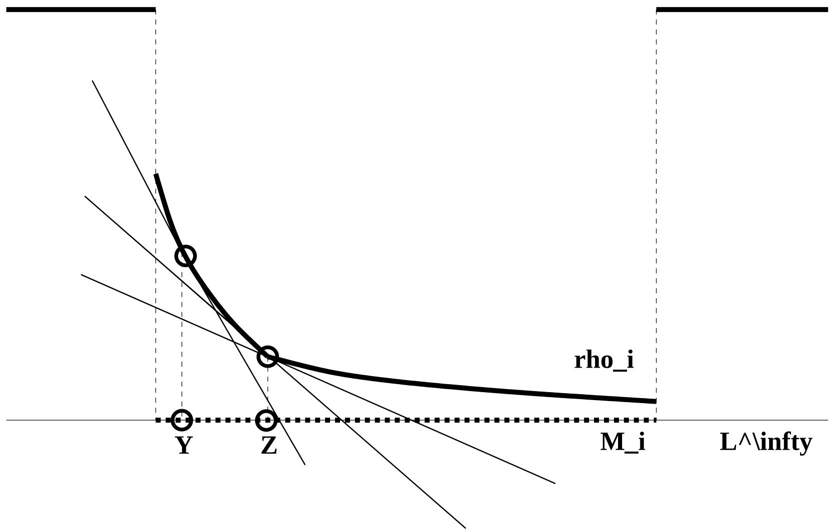
*where* $\mathcal{P} := \{\mu \in (L^\infty)^* \mid \langle \mu, 1 \rangle = 1,\, \mu \ge 0\}$ *= set of* **pricing rules**.

# Sub-Gradient

Let $f : L^\infty \to \mathbb{R} \cup \{+\infty\}$ be convex. The **sub-gradient** is

$$\partial f(\xi) := \{\nu \in (L^\infty)^* \mid f(\eta) \geq f(\xi) + \langle \nu, \eta - \xi \rangle \ \forall \eta \in L^\infty\}$$

Example: $f(\xi) = \rho_i^{M_i}(\xi) := \begin{cases} \rho_i(\xi), & \xi \in M_i \\ +\infty, & \text{else,} \end{cases}$



## Technical Results

**Proposition 2.** $\rho_i^{M_i} : L^\infty \to \mathbb{R} \cup \{+\infty\}$ *is convex, cash invariant,* $\sigma(L^\infty, (L^\infty)^*)$*-lower semi-continuous and can be represented by*

$$\rho_i^{M_i}(\xi) = \max_{\mu \in \mathcal{P}} \left( \langle \mu, -\xi \rangle - \alpha_i^{M_i}(\mu) \right), \quad \forall \xi \in M_i$$

*with "penalty function"* $\alpha_i^{M_i}(\mu) := \sup_{\xi \in M_i} (\langle \mu, -\xi \rangle - \rho_i(\xi))$.

*Moreover,*

$$\inf_{\langle \mu, \eta \rangle \leq \langle \mu, \xi \rangle} \rho_i^{M_i}(\eta) = \langle \mu, -\xi \rangle - \alpha_i^{M_i}(\mu), \quad \forall \mu \in \mathcal{P},\ \xi \in M_i.$$

*And the following statements are equivalent:*

1. $-\mu \in \partial \rho_i^{M_i}(\xi)$
2. $\rho_i^{M_i}(\xi) = \langle \mu, -\xi \rangle - \alpha_i^{M_i}(\mu)$
3. $\inf_{\langle \mu, \eta \rangle \leq \langle \mu, \xi \rangle} \rho_i^{M_i}(\eta) = \rho_i^{M_i}(\xi) \qquad$ *(for $\mu \in \mathcal{P}$)*

**Corollary 3.** $\partial \rho_i^{M_i}(\xi) \cap (-\mathcal{P}) \neq \emptyset$, $\forall \xi \in M_i$.

## Equilibrium and Optimality

An allocation $\xi_1, \ldots, \xi_n$ is **attainable** if $\xi_i \in M_i$ and $\sum_i \xi_i \leq X$.

An attainable allocation $\xi_1, \ldots, \xi_n$ is Pareto **optimal** if

$$\rho_i(\eta_i) \leq \rho_i(\xi_i) \quad \forall i \quad \Rightarrow \quad \rho_i(\eta_i) = \rho_i(\xi_i) \quad \forall i$$

for every attainable allocation $\eta_1, \ldots, \eta_n$.

An attainable allocation $\xi_1, \ldots, \xi_n$ together with a pricing rule $\mu \in \mathcal{P}$ is an Arrow–Debreu **equilibrium** if

$$\langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle \quad \text{and} \quad \rho_i(\xi_i) = \inf_{\langle \mu, \eta \rangle \leq \langle \mu, X_i \rangle,\, \eta \in M_i} \rho_i(\eta) \quad \forall i.$$

## Optimal capital and risk transfer is Pareto optimal

Let $\xi_1, \ldots, \xi_n$ be an attainable allocation, $\xi_0 := X - \sum_i \xi_i \geq 0$. The following are equivalent:

1. $\square_i^{(M_i)} \rho_i(X) = \sum_{i=1}^n \rho_i(\xi_i)$ **(original optimization problem)**

2. $\xi_1, \ldots, \xi_n$ is Pareto optimal

3. $\langle \mu, \xi_0 \rangle = 0$ **(price clearing)** and $-\mu \in \partial\rho_1^{M_1}(\xi_1) \cap \cdots \cap \partial\rho_n^{M_n}(\xi_n)$ for some $\mu \in \mathcal{P}$. **(first order condition)**

## Welfare Theorems

$\xi_1, \ldots, \xi_n, \mu$ is an equilibrium if and only if

1. $\xi_1, \ldots, \xi_n$ is optimal and
2. $-\mu \in \partial \Box_i^{(M_i)} \rho_i(X)$ (**optimal pricing rules**) and
3. $\langle \mu, \xi_i \rangle = \langle \mu, X_i \rangle$ for all $i$.

Moreover, if $\xi_1, \ldots, \xi_n$ is optimal and $-\mu \in \partial \Box_i^{(M_i)} \rho_i(X)$ then

$$\xi_1 + \langle \mu, X_1 - \xi_1 \rangle, \ldots, \xi_n + \langle \mu, X_n - \xi_n \rangle$$

is an equilibrium.

## Game Theoretic View

Allocation $(k_1, \ldots, k_n) \in \mathbb{R}^n$ is in the **core** of the game with characteristic function $\{1, \ldots, n\} \supset S \mapsto \Box_{\mathbf{i \in S}}^{(M_i)} \rho_i \left(\sum_{i \in S} X_i\right)$ if

$$\sum_{i=1}^{n} k_i = \Box_i^{(M_i)} \rho_i(X) \quad \text{and}$$

$$\sum_{i \in S} k_i \le \Box_{\mathbf{i \in S}}^{(M_i)} \rho_i \left( \sum_{i \in S} X_i \right) \qquad \forall S \subset \{1, \ldots, n\}.$$

**Proposition 4.** *Let $\xi_1, \ldots, \xi_n, \mu$ be an equilibrium. Then*

$$(\rho_1(\xi_1), \ldots, \rho_n(\xi_n)) = (\langle \mu, -X_1 \rangle - \alpha_1^{M_1}(\mu), \ldots, \langle \mu, -X_n \rangle - \alpha_n^{M_n}(\mu))$$

*lies in the core of this game.*

## Approximate Equilibrium

Optimal allocations do not always exist! BUT:

Let $-\mu \in \partial \Box_i^{(M_i)} \rho_i(X)$ $(\neq \emptyset, \subset (-\mathcal{P}))$ and $\epsilon > 0$. Then there exists an attainable allocation $\xi_1, \ldots, \xi_n$ such that:

1. $\Box_i^{(M_i)} \rho_i(X) \geq \sum_{i=1}^n \rho_i(\xi_i) - n\epsilon$ **("$\epsilon$-optimality")** and

2. $\langle \mu, -X_i \rangle - \alpha_i^{M_i}(\mu) \geq \rho_i(\xi_i) - \epsilon$ **("$\epsilon$-equilibrium")** and

3. $\langle \mu, X_i \rangle - \epsilon \leq \langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle$ for all $i$.

Practical limitations: optimal pricing rules $\partial \Box_i^{(M_i)} \rho_i(X)$ may be difficult to find

# Example for Non-Existence of Optimal Allocations

Jouini, Schachermayer, Touzi (05): $n = 2$, $M_i = L^\infty$

Let $D \geq 0$, $\mathbb{E}[D] = 1$, $\mathbb{P}[D = 0] > 0$, $\mathbb{E}[D \log D] < \infty$, and define

$$\rho_1(\xi) := \log \mathbb{E}[\exp(-\xi)], \quad \rho_2(\xi) := -\mathbb{E}[D\xi].$$

Variational principle for relative entropy:

$$\log \mathbb{E}[e^{-\xi}] + \mathbb{E}[D\xi] \geq -\mathbb{E}[D \log D] \quad \forall e^{-\xi} \in L^1; \quad \text{“}=\text{”} \Leftrightarrow e^{-\xi} = D$$

$\Rightarrow \rho_1(\xi) + \rho_2(X - \xi) \geq \mathbb{E}[DX] - \mathbb{E}[D \log D] > -\infty \quad \forall \xi \in L^\infty$

$\Rightarrow \rho_1 \Box \rho_2(X) \geq \mathbb{E}[DX] - \mathbb{E}[D \log D] > -\infty$ (standing assumption satisfied)

But $\xi = -\log D \notin L^\infty$. Hence optimal allocation does not exist in $L^\infty$.

# Special Case

Let $Z_0, \ldots, Z_m \in L^\infty$ be linearly independent, with $Z_0 \equiv 1$ ("cash").

**Assumptions:**

- $X_i \notin \text{span}(Z_0, \ldots, Z_m)$ for all $i$
- $M_i = \left\{ X_i + \sum_{j=0}^m \lambda_j Z_j \mid \lambda_j \in \mathbb{R} \right\}$ finite-dimensional affine spaces
- $v_i(\lambda_0, \ldots, \lambda_m) := \rho_i(X_i + \sum_{j=0}^m \lambda_j Z_j)$ differentiable in $\lambda_j$

Then

1. $v_i(\lambda_0, \ldots, \lambda_m) = v_i(0, \lambda_1, \ldots, \lambda_m) - \lambda_0 =: u_i(\lambda_1, \ldots, \lambda_m) - \lambda_0$
2. $\nu \in \partial \rho_i^{M_i}(X_i + \sum_j \lambda_j Z_j) \Leftrightarrow \langle \nu, 1 \rangle = -1$ and $\langle \nu, Z_j \rangle = \partial_{\lambda_j} u_i(\lambda_1, \ldots, \lambda_m)\ \forall j \geq 1$

$\rightarrow$ Equilibrium pricing rule (if it exists) does not interfere with any a priori values of $X_1, \ldots, X_n$!

The clearing condition

$$\sum_{i=1}^{n}\left(X_i+\sum_{j=0}^{m}\lambda_{ij}Z_j\right)\leq X \Longleftrightarrow \sum_{j=0}^{m}\left(\sum_{i=1}^{n}\lambda_{ij}\right)Z_j\leq 0$$

is always satisfied (“and vice versa”) for

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\lambda_{ij}\right)Z_j-\underbrace{\operatorname{ess\,sup}\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\lambda_{ij}Z_j\right)}_{\text{net cash}}Z_0$$

$\rightarrow$ $(m\times n)$-dimensional optimization problem

$$\square_i^{(M_i)}\rho_i(X)=\inf_{\lambda_{ij}}\left(\sum_{j=1}^{m}\sum_{i=1}^{n}u_i(\lambda_{i1},\ldots,\lambda_{im})+\operatorname{ess\,sup}\left(\sum_{j=1}^{m}\sum_{i=1}^{n}\lambda_{ij}Z_j\right)\right)$$

# Characterization of Optimum

**Corollary.** The optimum is attained:

$$\square_i^{(M_i)} \rho_i(X) = \sum_{j=1}^{m} \sum_{i=1}^{n} u_i(\lambda_{i1}^*, \ldots, \lambda_{im}^*) + \operatorname{ess\,sup} \left( \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_{ij}^* Z_j \right)$$

if and only if

1. $\partial_{\lambda_j} u_i(\lambda_{i1}^*, \ldots, \lambda_{im}^*) = \partial_{\lambda_j} u_1(\lambda_{11}^*, \ldots, \lambda_{1m}^*) =: \langle \mu, -Z_j \rangle \ \forall i, j$ and

2. $\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_{ij}^* \langle \mu, Z_j \rangle = \operatorname{ess\,sup} \left( \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_{ij}^* Z_j \right)$ **(price clearing)**

An equilibrium allocation is then obtained by

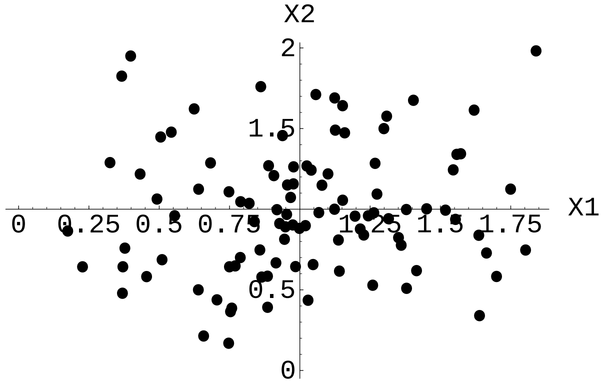
$$X_i + \sum_{j=1}^{m} \lambda_{ij}^* Z_j - \sum_{j=1}^{m} \lambda_{ij}^* \langle \mu, Z_j \rangle$$

and $\langle \mu, Z_j \rangle$ are the **fair prices** for the risk and capital transfers $Z_j$

## Concrete Example

$m = n = 2$ insurance units. $Z_i = (C_i - m_i)^+$ surplus participation.

$(C_1, C_2) =$ random (100pt) sample of $N\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \sigma^2 \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}\right)$




Assumption: available $(= C_i(0) = 1) =$ required capital:

$$0 = \text{ES}\,(C_i) = \sigma \text{ES}\,(N(0,1)) - 1 \Leftrightarrow \sigma = \frac{1}{\text{ES}\,(N(0,1))} = 0.375204,$$

for confidence level $\alpha = 0.01$.

$\text{ES}(C_1) = -0.175$, $\text{ES}(C_2) = -0.171$, $\text{ES}(C_1 + C_2) = -0.850$.

Minimum capital $m_i := 0.4 \times (1 + \text{ES}(C_i))$ (=SST risk margin)

$m_1 = 0.330$, $m_2 = 0.332$.

Optimal allocation $(\lambda_{11}^*, \lambda_{12}^*) = (-0.58, 0.75)$, $(\lambda_{21}^*, \lambda_{22}^*) = (0.50, -0.81)$.

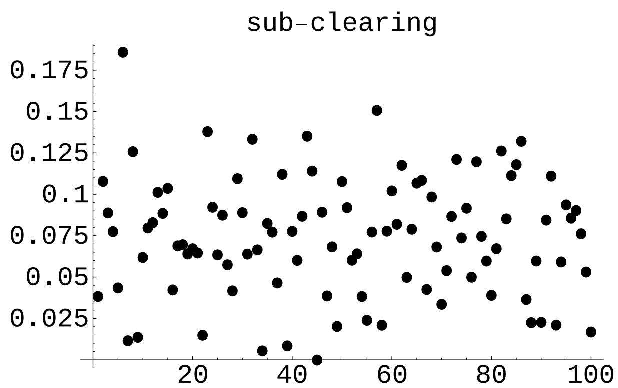
net cash flow = 0.01,

fair values: $\langle \mu, (C_1 - m_1)^+ \rangle = 0.04$, $\langle \mu, (C_2 - m_2)^+ \rangle = 0.15$.

$$\text{ES}\left(C_1 - 0.58\left((C_1 - m_1)^+ - 0.04\right) + 0.75\left((C_2 - m_2)^+ - 0.15\right)\right) = -0.37$$

$$\text{ES}\left(C_2 + 0.50\left((C_1 - m_1)^+ - 0.04\right) - 0.81\left((C_2 - m_2)^+ - 0.15\right)\right) = -0.48$$

sum $= -0.85 \Rightarrow$ full diversification effect (in general not)



# Summary

We propose an equilibrium approach to valuate and measure contingent capital and risk transfers between insurance units in order to account for diversification benefits.

The available capital does not change (equilibrium) while the maximal reduction of required capital is achieved (optimum).

The first example with surplus participations $Z_i = (C_i - m_i)^+$ can be generalized. Results are available for arbitrary convex and cash invariant sets $M_i \subset L^\infty$ of capital and risk transfers (D.F. and M.Kupper: "Equilibrium and optimality for monetary utility functions under constraints")