QMF 2005, Sydney

Finite Horizon Optimal Investment with Transaction Costs:

A Parabolic Double Obstacle Problem

Min Dai

Dept of Math
National University of Singapore

Joint with Fahuai Yi.

Literature Review

- Merton (1971) first studied the optimal investment and consumption problem without transaction costs using continuous stochastic models.

- Magill and Constantinides (1976) introduced (proportional) transaction costs to Merton's model.

- Davis and Norman (1990) and Shreve and Soner (1994) studied the problem with transaction costs for an infinite time horizon.

- Lui and Loewenstein (2002) considered finite horizon optimal investment problem with transaction costs.

Contents

Asset market

Two assets

$$
\begin{aligned}
dP_{0t} &= rP_{0t}dt, \\
dP_{1t} &= P_{1t}\left[\alpha dt + \sigma d\mathcal{B}_t\right],
\end{aligned}
$$

where $\alpha > r$

An investor holds X_t and Y_t in bank and stock, respectively.

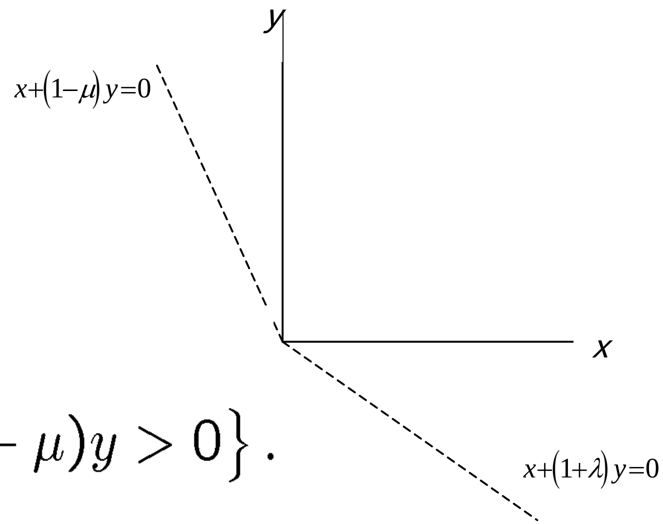
$$
\begin{aligned}
dX_t &= rX_t dt - (1+\lambda)dL_t + (1-\mu)dM_t \\
dY_t &= \alpha Y_t dt + \sigma Y_t d\mathcal{B}_t + dL_t - dM_t,
\end{aligned}
$$

L_t: cumulative dollar value for buying stock
M_t: cumulative dollar value for selling stock
$\lambda \in [0, \infty)$: proportion of transaction costs for purchase
$\mu \in [0, 1)$: proportion of transaction costs for sale

Solvency region

The investor's net wealth in monetary terms

$$W_t = \begin{cases} X_t + (1-\mu)Y_t & \text{if } Y_t \geq 0, \\ X_t + (1+\lambda)Y_t & \text{if } Y_t < 0. \end{cases}$$



Define the solvency region

$$\mathscr{S} = \left\{(x,y) \in R^2 : x + (1+\lambda)y > 0,\ x + (1-\mu)y > 0\right\}.$$

An adimssible investment strategy: (L_t, M_t)

$A(x_0, y_0)$: the set of admissible investment strategies.

Optimal problem and value function

The investor's problem:

$$\sup_{(L_t,M_t)\in A(x_0,y_0)} E_0^{x_0,y_0}\left[U(W_T)\right]$$

where the utility function

$$U(W) = \frac{W^\gamma}{\gamma} \qquad \text{if } \gamma < 1,\ \gamma \neq 0,$$

Value function

$$\varphi(x_t, y_t, t) = \sup_{(L_t,M_t)\in A(x_t,y_t)} E_t^{x_t,y_t}\left[U(W_T)\right],\ (x_t, y_t) \in \mathscr{S},\ t \in [0, T)$$

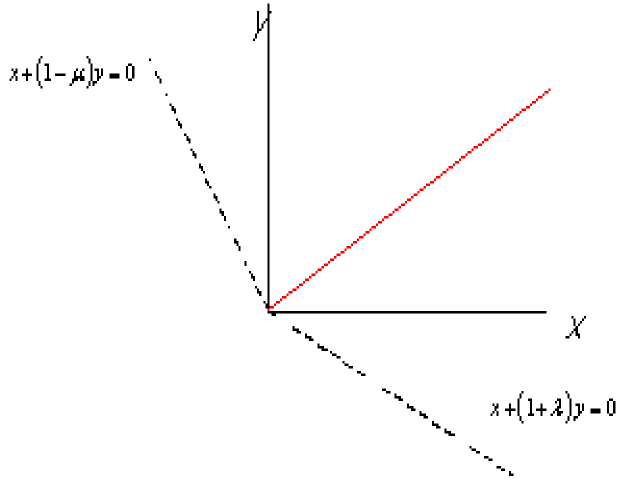
No transaction costs (Merton)

$$\varphi(x,y,t) = e^{\gamma\left(r+\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}\right)(T-t)}\frac{(x+y)^\gamma}{\gamma} \qquad \text{if } \gamma < 1,\ \gamma \neq 0.$$

Optimal policy

$$\frac{x}{y} = -\frac{\alpha - r - (1-\gamma)\sigma^2}{\alpha - r} \triangleq x_M.$$

Here x_M is called "Merton line".

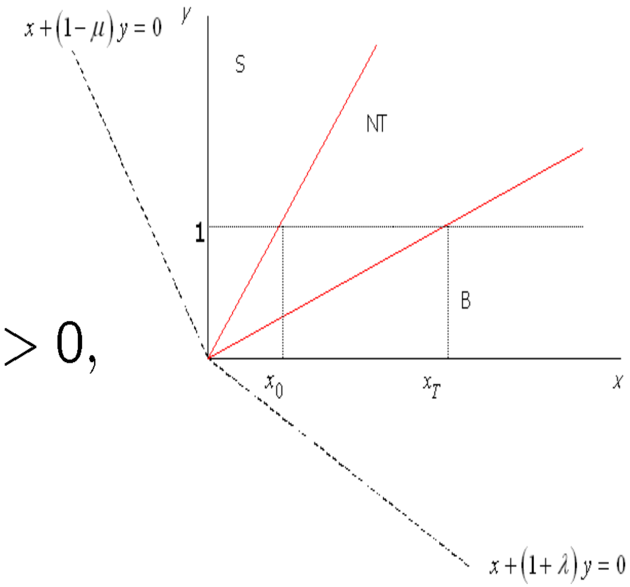


Proportional transaction costs

$$\min\{-\varphi_t - \mathscr{L}\varphi, -(1-\mu)\varphi_x + \varphi_y, (1+\lambda)\varphi_x - \varphi_y\} = 0,$$
$$(x, y) \in \mathscr{S},\ \ t \in [0, T)$$

$$\varphi(x, y, T) = \begin{cases} U\left(x + (1-\mu)y\right) & \text{if } y \geq 0, \\ U\left(x + (1+\lambda)y\right) & \text{if } y < 0, \end{cases}$$

where $\mathscr{L}\varphi = \frac{1}{2}\sigma^2 y^2 \varphi_{yy} + \alpha y \varphi_y + r x \varphi_x$.



Due to homotheticity of utility function, for any $\rho > 0$,

$$\varphi(\rho x, \rho y, t) = \rho^\gamma \varphi(x, y, t) \quad \text{if } \gamma < 1,\ \gamma \neq 0.$$

Let $V(x, t) = \varphi(x, 1, t)$.

Reduced problem

$$\begin{cases} \min\left\{-V_t - \mathcal{L}_1 V, -(x+1-\mu)V_x + \gamma V, (x+1+\lambda)V_x - \gamma V\right\} = 0, \\ V(x,T) = \frac{1}{\gamma}(x+1-\mu)^\gamma \quad \text{in } \Omega, \end{cases}$$

where $\Omega = (-(1-\mu), +\infty) \times [0,T)$,

$$\mathcal{L}_1 V = \frac{1}{2}\sigma^2 x^2 V_{xx} + \beta_2 x V_x + \beta_1 V$$

with $\beta_1 = \gamma\left(\alpha - \frac{1}{2}\sigma^2(1-\gamma)\right)$ and $\beta_2 = -\left(\alpha - r - \sigma^2(1-\gamma)\right)$.

A parabolic double obstacle problem

$$\begin{cases} -u_t - \mathcal{L}u = 0 & \text{if } \frac{1}{x+1+\lambda} < u < \frac{1}{x+1-\mu}, \\ -u_t - \mathcal{L}u \geq 0 & \text{if } u = \frac{1}{x+1+\lambda}, \qquad\qquad -(1-\mu) < x < \infty, \\ -u_t - \mathcal{L}u \leq 0 & \text{if } u = \frac{1}{x+1-\mu}, \qquad\qquad 0 \leq t < T \\ u(x,T) = \frac{1}{x+1-\mu} \end{cases}$$

where

$$\mathcal{L}u = \frac{1}{2}\sigma^2 x^2 u_{xx} - \left(\alpha - r - (2-\gamma)\sigma^2\right) x u_x - (\alpha - r - (1-\gamma)\sigma^2) u + \gamma\sigma^2 \left(x^2 u u_x + x u^2\right)$$

Proposition $u(x,t) \in W_p^{2,1},\ 1 < p < +\infty$ .Moreover,

$$u_t(x,t) \geq 0.$$

Three regions

$$\mathbf{SR} = \left\{(x,t) : u(x,t) = \frac{1}{x+1-\mu}\right\},$$

$$\mathbf{BR} = \left\{(x,t) : u(x,t) = \frac{1}{x+1+\lambda}\right\},$$

$$\mathbf{NT} = \left\{(x,t) : \frac{1}{x+1+\lambda} < u(x,t) < \frac{1}{x+1-\mu}\right\}.$$

Proposition

(i) **SR**$\subset \{(x,t) : x \leq (1-\mu)x_M\}$;

(ii) **BR**$\subset \{(x,t) : x \geq (1+\lambda)x_M\}$.

Free boundary (1): selling policy

Theorem There is a continuous, monotonically increasing function $x_s^*(t)$, $t \in [0,T)$, such that

$$\mathbf{SR} = \{(x,t) : x \le x_s^*(t),\ t \in [0,T)\}\,.$$

Moreover,

(i) $x_s^*(t)$ is strictly monotone if $\alpha - r - (1-\gamma)\sigma^2 \neq 0$, and

$$x_s^*(t) \equiv 0 \qquad \text{if } \alpha - r - (1-\gamma)\sigma^2 = 0;$$

(ii) $x_s^*(t) \in C^\infty[0,T)$,

$$x_s^*(T^-) \triangleq \lim_{t \to T^-} x_s^*(t) = (1-\mu)x_M\,,$$

$$\lim_{T-t \to +\infty} x_s^*(T-t) = x_{s,\infty}^*.$$

Free boundary (2): buying policy

Theorem Let $t_0 = T - \frac{1}{\alpha - r}\log(\frac{1+\lambda}{1-\mu})$.

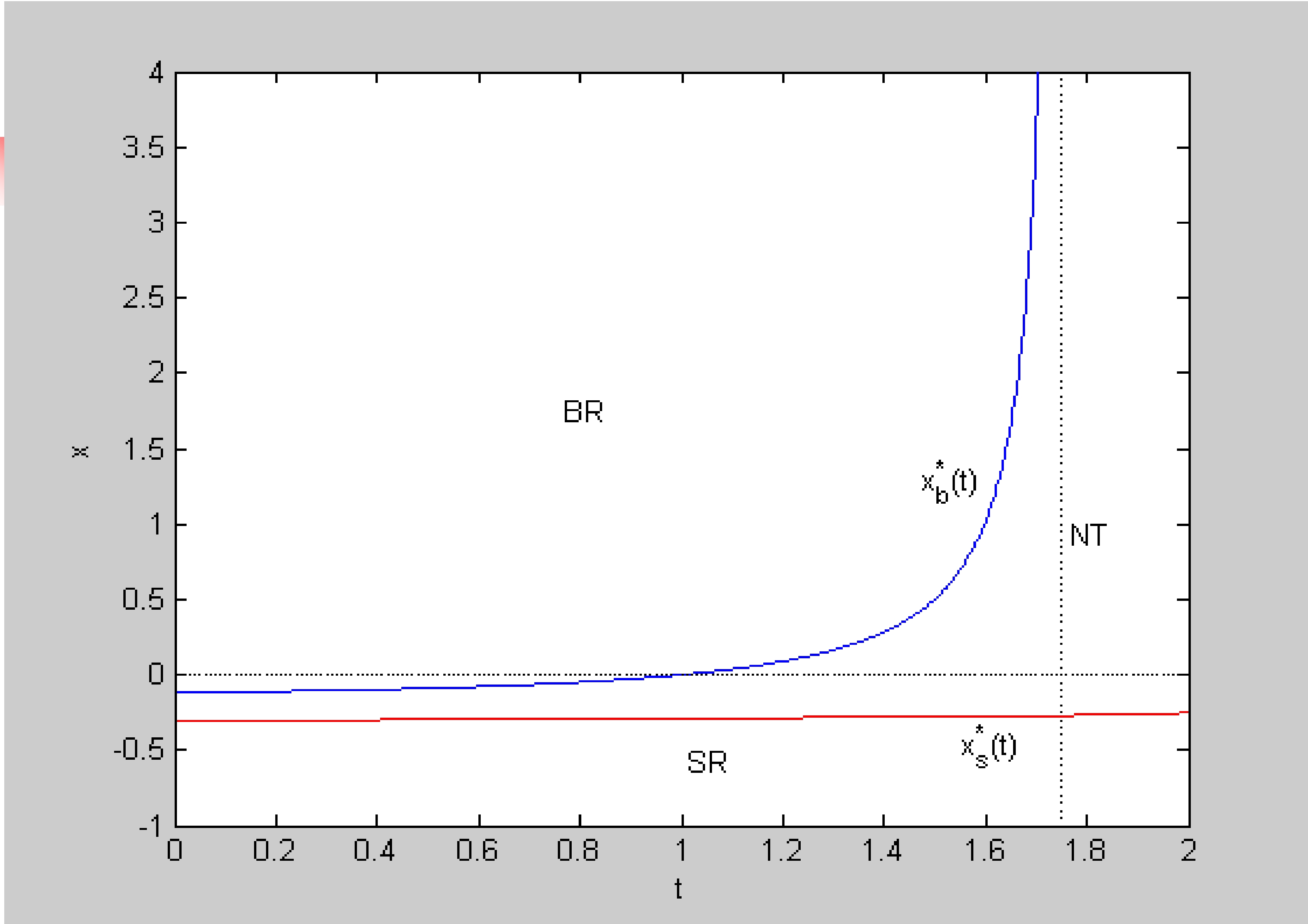
(i) When $t \in [t_0, T)$, **BR**$= \emptyset$. When $t \in [0, t_0)$, there is a strictly monotonically increasing function $x_b^*(t)$ s.t.

$$\mathbf{BR} = \{(x,t) : x \geq x_b^*(t),\ 0 \leq t < t_0\}.$$

$$\lim_{t \to t_0^-} x_b^*(t) = +\infty,$$

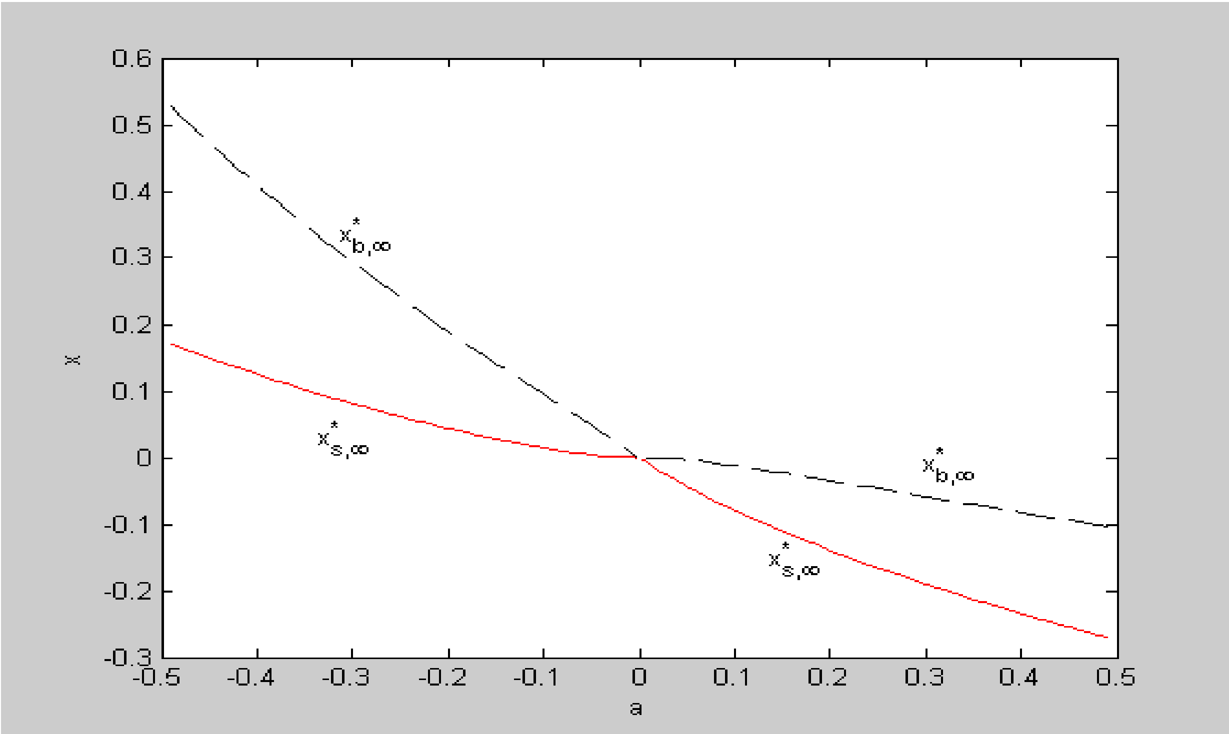
$$\lim_{T-t \to +\infty} x_b^*(T-t) = x_{b,\infty}^*.$$

(iii) When $\alpha - r - (1-\gamma)\sigma^2 \geq 0$, $x_b^* > 0$; When $\alpha - r - (1-\gamma)\sigma^2 > 0$, $x_b^*(t_1) = 0$, where $t_1 = T - \frac{1}{\alpha - r - (1-\gamma)\sigma^2}\log\frac{1+\lambda}{1-\mu}$.



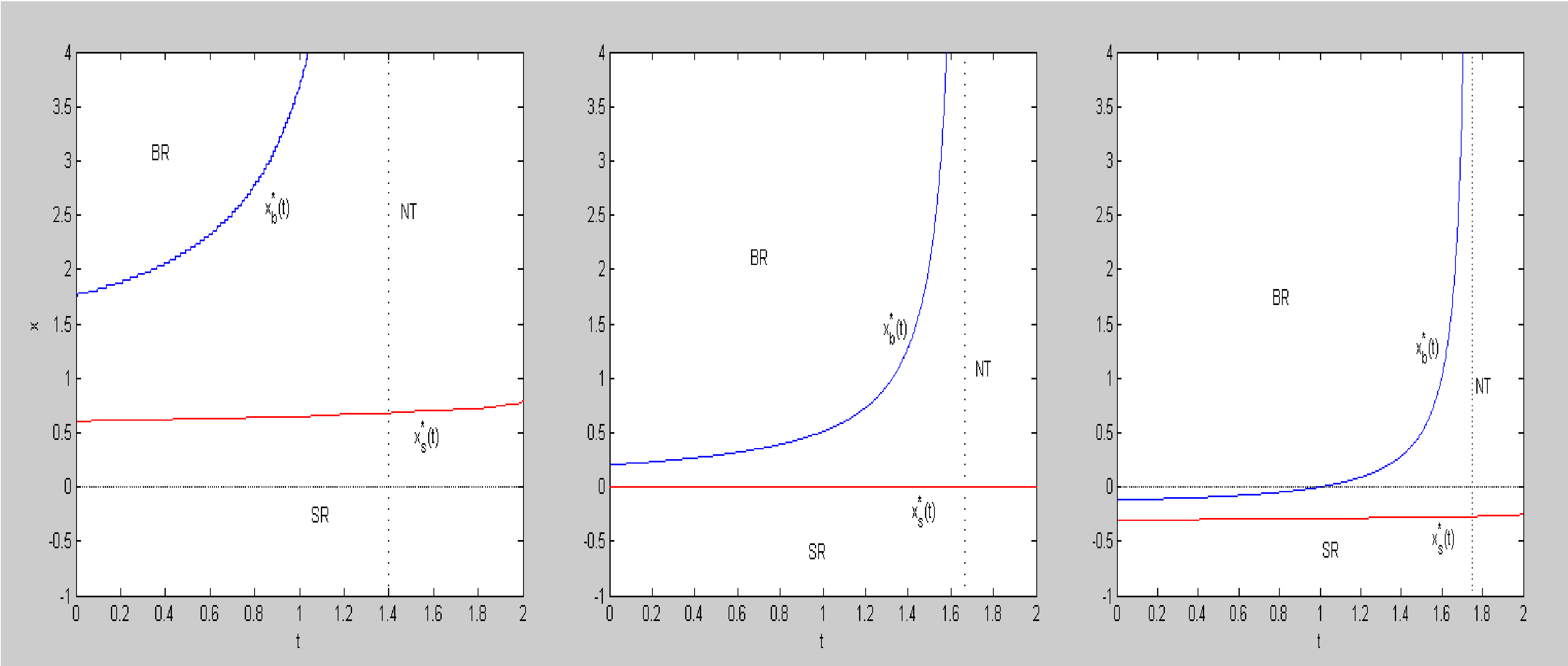
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
x
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t
BR
SR
NT
x*b(t)
x*s(t)

Asymptotic behavior as time to expiry tends to infinity



$$x^*_{s,\infty} = -\frac{a}{a+k}(1-\mu), \;\; x^*_{b,\infty} = -\frac{a}{a+\frac{k}{k-1}}(1+\lambda),$$

where $a = \frac{\alpha - r - (1-\gamma)\sigma^2}{\frac{1}{2}(1-\gamma)\sigma^2}$



$$\alpha - r < (1-\gamma)\sigma^2 \qquad \alpha - r = (1-\gamma)\sigma^2 \qquad \alpha - r > (1-\gamma)\sigma^2$$

Numerical methods

Denote

$$\bar{r} = \alpha - r - (1-\gamma)\sigma^2 \text{ and } \bar{q} = 2(\alpha - r) - (3 - 2\gamma)\sigma^2.$$

Then

$$\mathcal{L}u = \frac{1}{2}\sigma^2 x^2 u_{xx} + (\bar{r} - \bar{q})\, x u_x - \bar{r}u + \gamma\sigma^2 x u\,(x u_x + u)$$

Conclusion

- Derivation of the double obstacle problem
- Completely characterize the optimal investment policy
- An efficient numerical algorithm

Future work

- Consider multi-risky assets, consumption, fixed proportional transaction costs, etc.