



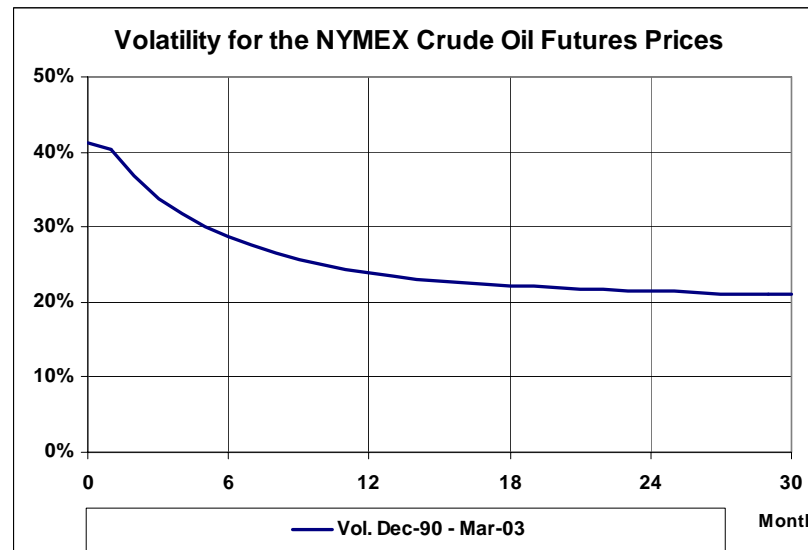
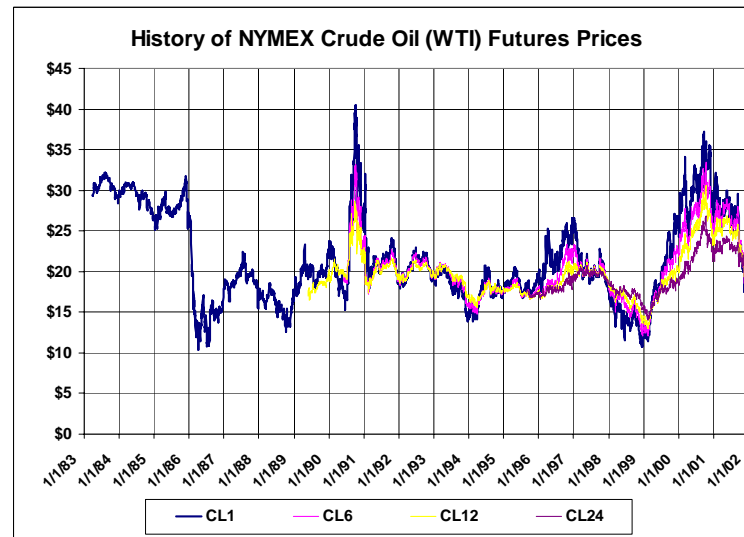
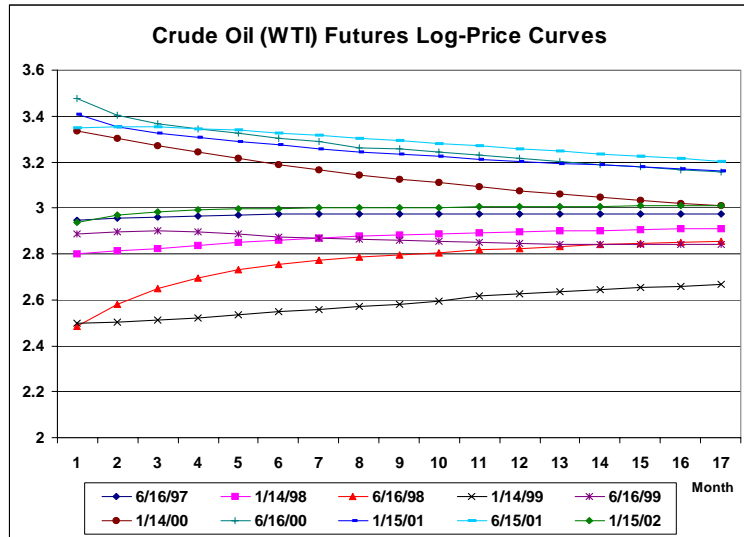
# **Classes of Multifactor HJM-Type Futures Curve Models**

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# Statistical Properties of the Crude Oil Futures Curves



## Empirical Mean-Reverting “Modified Principal Component” Model

Suppose the dynamics of the futures log-price curve is described by a linear combination of “smooth” driver functions (MPCs),  $\eta_i(\tau)$ , with the stochastic mean-reverting coefficients (state variables)  $z_i(t)$ :

$$X(t, \tau) = \sum_{i=1}^N \eta_i(\tau) z_i(t) = \bar{\eta}(\tau)' \bar{z}(t)$$

Here the driver vector  $\bar{z}(t)$  follows a multidimensional Gaussian mean reversion system with some matrix of mean reversion speeds,  $\Lambda$ , and the covariance matrix of instantaneous returns,  $C$ :

$$d\bar{z}(t) = \Lambda(\bar{\mu} - \bar{z}(t))dt + B d\bar{w}, \quad C = B' B$$

Question: What stochastic system with respect to the points on the futures log-price curve corresponds to this model?

**Proposition 1.** *Let the dynamics of the futures log-price curve is described by the linear combination of “smooth” driver functions,  $\eta_i(\tau)$ ,  $\tau \geq 0$ , with the stochastic coefficients (state variables)  $\bar{z}(t)$  following Gaussian mean reversion system above. Then the dynamics of the entire futures log-price curve  $X(t, \tau)$  is described by the following infinite dimensional linear stochastic system*

$$dX(t, \tau) = \int_0^{\infty} K(\tau, s) [\theta(s) - X(t, s)] ds dt + \bar{\sigma}(\tau)' d\bar{w}, \quad t \geq 0, \quad \tau \geq 0,$$

with the “smooth” integral kernel, volatility and mean functions are of the form:

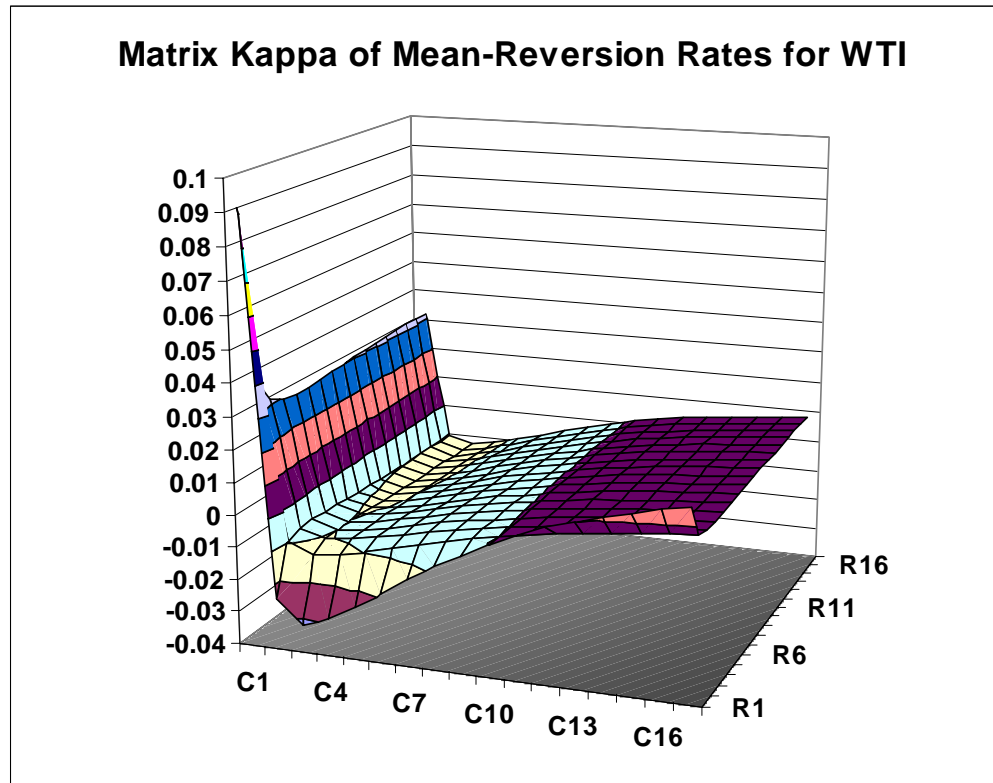
$$K(\tau, s) = \bar{\eta}(\tau)' \Lambda M^{-1} \bar{\eta}(s), \quad \bar{\sigma}(\tau) = B' \bar{\eta}(\tau), \quad \theta(s) = \bar{\eta}(s)' \bar{\mu},$$

$$M = \int_0^{\infty} \bar{\eta}(s) \bar{\eta}(s)' ds$$

Full square matrix of mean reversion speeds  $\mathbf{K}$  in Gaussian mean reversion system above corresponds to finite-dimensional discretization of the smooth kernel  $K(\tau, s)$ . For example, if the driver functions are exponential functions,

$$\eta_i(\tau) = \exp(-\lambda_i \tau), \quad \lambda_i > 0,$$

then the matrix  $M$  is simply a matrix with the elements  $M_{ij} = (\lambda_i + \lambda_j)^{-1}$ .



The matrix  $\mathbf{K}$  of mean reversion rates is estimated by the method of Modified Principal Components [Levin, 2002] that allows to fit into three covariance matrices of infinitesimal returns, intermediate lag returns and long-term (stationary) futures log-prices using two eigenvalue decompositions and Lyapunov equation.

## Method of Modified Principal Components

**Proposition 2.** *The parameters of the mean reversion system above are uniquely and constructively defined by the mean vector  $\bar{\theta}$ , covariance matrix  $R$  of the infinitesimal returns, covariance matrix  $Q_\infty$  of the long-term (stationary) risk factor distribution, and covariance matrix  $Q_\delta$  (without a diagonal) of some medium-term  $\delta > 0$  risk factor return distribution. The input matrices,  $R$  and  $Q_\infty$ , that uniquely define the short-term and long-term Gaussian risk factor distributions are independent. The input medium-term covariance matrix  $Q_\delta$ , given the matrices  $C$  and  $Q_\infty$ , may deviate within some boundaries that ensure positive eigenvalues of the mean reversion speed matrix  $K$ .*

$$\tilde{B} = HS, \quad R = (HS)(SH'), \quad H' = H^{-1}, \quad S = \text{diag}(s_1, \dots, s_N), \quad s_1 \geq \dots \geq s_N > 0$$

$$\tilde{Q}_\infty = \tilde{B}^{-1} Q_\infty (\tilde{B}^{-1})' = VDV', \quad V' = V^{-1}, \quad D = \text{diag}(d_1, \dots, d_N), \quad d_i > 0$$

$$\Omega = V' \tilde{B}^{-1} K \tilde{B} V, \quad \Omega = \frac{1}{2} (I + W) D^{-1}, \quad W' = -W$$

$$\tilde{Q}_\delta = V' \tilde{B}^{-1} Q_\delta (\tilde{B}^{-1})' V, \quad \Omega \tilde{Q}_\delta + \tilde{Q}_\delta \Omega' = I - e^{-\Omega \delta} e^{-\Omega' \delta}$$

## A Class of “Stationary” HJM-Type Futures Curve Models

Let  $F(t, \tau) > 0$  be the price at time  $t \geq 0$  of a futures contract for delivery at maturity time  $T \geq t$ . Time  $t = 0$  is viewed as the current time. Consider a continuous futures curve model, i.e. assume that the futures contracts are traded for all maturities  $T \geq t$ .

Suppose the stochastic process for the entire futures curve is driven by  $M \geq 1$  independent standard Brownian motions,  $\bar{W}(t) = (W_1(t), \dots, W_M(t))'$ , and this process satisfies usual regularity conditions. As shown by Harrison and Kreps and Harrison and Pliska the absence of arbitrage in the economy under very general conditions implies the existence of equivalent martingale measure under which the discounted asset prices follow a martingale. The instantaneous return on the futures contracts for all maturities is equal to zero, because futures contracts involve zero investment. This means that for all fixed maturities  $T > 0$ , the stochastic differential equation for the futures prices has zero drift term under the equivalent martingale measure:

$$dF(t, T) = \sum_{j=1}^M \tilde{\sigma}_j^F(t, T) dw_j(t)$$

The choice of the volatility functions completely determines the particular model. Usually, time-homogeneous (i.e. dependent only on the time to maturity,  $\tau = T - t$ ) volatility functions are considered. The class of time-homogeneous volatility functions is very important for the long-term simulation models, because it corresponds to “stationary” models and allows for historical (statistical) calibration based on the futures price historical time series.

We consider a  $\Phi$ -affine class of models corresponding to an arbitrary (not only exponential) one-to-one non-linear static transformation,  $\Phi(x)$ , of a linear combination of the time-homogeneous driver functions,  $\eta_j(T-t)$ , with the stochastic coefficients (state variables),  $z_j(t)$ , that follow a general mean reversion Gaussian system with constant parameters:

$$F(t, T) = \Phi(X(t, T)), \quad \Phi'(x) \neq 0$$

$$X(t, T) = \sum_{j=1}^M \eta_j(T-t) z_j(t) + \Psi(t, T) = \bar{\eta}(T-t)' \bar{z}(t) + \Psi(t, T)$$

$$d\bar{z}(t) = -\mathbf{K} \bar{z}(t) dt + B d\bar{w}, \quad \bar{z}(t) = (z_1(t), \dots, z_M(t))', \quad M \geq 1$$

$$\bar{z}(0) = \bar{0}$$

$$X(0, T) = \Psi(0, T) = \Phi^{-1}(F(0, T)) = X_0(T)$$

Non-linear function  $\Phi(x)$  is assumed static (time and maturity-independent), one-to-one ( $\Phi'(x) \neq 0$ ), sufficient number of times differentiable transformation. This condition allows for inverse transformation  $X(t, T) = \Phi^{-1}(F(t, T))$ , and then for statistical analysis of the Gaussian time-series.

Examples of static transformations are exponential (“Black-Karasinski-type” model) and linear (“Vasicek-type” model) transformations. It is assumed that a general  $M \times M$  matrix  $K$  with the mean reversion rates has non-negative real parts for the corresponding (possibly complex) characteristic values. Positive characteristic values correspond to a stable mean reversion system. Zero values are allowed. They correspond to co-integrated stochastic systems with non-zero Wiener components (for example, one-factor Black’s model with the exponential transformation and flat volatility function). Matrix  $B$  is assumed a non-singular arbitrary matrix of size  $M \times M$ . The deterministic function  $\Psi(t, T)$  is equal to the  $\Phi$ -inverse of the initial futures curve at time zero. It is also shown below that, in the case of mean reversion system, the function  $\Psi(t, T) = \Psi(t, t + \tau)$  has a “stationary” limit at  $t = \infty$  dependent only on the time to maturity  $\tau = T - t$ .

**Proposition 3.** *The futures price curve dynamics under the risk neutral measure for the model above is described by the stochastic differential equation of the form*

$$dF(t, T) = \Phi'(\Phi^{-1}(F(t, T))) \sum_{j=1}^M \sigma_j (T - t) dw_j = \Phi'(\Phi^{-1}(F(t, T))) \bar{\sigma} (T - t)' d\bar{w}$$

*with zero drift and the time-homogeneous volatility functions,  $\bar{\sigma}(\tau)$ , expressed in terms of the driver functions,  $\bar{\eta}(\tau)$ , as  $\bar{\sigma}(\tau) = B' \bar{\eta}(\tau)$*

The following theorem completely characterizes the considered class of no-arbitrage futures curve models.

**Theorem 1.** *The necessary and sufficient conditions for the no-arbitrage dynamics of the futures price curve of the form above are as follows:*

a) *A one-to-one static transformation  $\Phi(x)$  is only of the form*

$$\Phi(x) = \pm \int e^{\frac{1}{2}as^2+bs+c} ds + k = \tilde{c} \int e^{\frac{1}{2}as^2+bs} ds + k$$

*with four constant parameters  $a, b, c,$  and  $k$  ( $\tilde{c} \neq 0$ ).*

b) *The time-homogenous driver functions,  $\bar{\eta}(\tau)$ , satisfy a (non-linear for  $a \neq 0$ ) system of ordinary differential equations (ODEs) with non-zero initial conditions:*

$$\begin{aligned} \bar{\eta}'(\tau) &= -\mathbf{K}'\bar{\eta}(\tau) + \frac{1}{2}a[\bar{\eta}(\tau)'BB'\bar{\eta}(\tau)]\bar{\eta}(\tau), \quad \tau > 0 \\ \bar{\eta}(0) &= \bar{\xi}, \quad \bar{\xi} \neq \bar{0}. \end{aligned}$$

c) *For all fixed  $T > 0$ , the function  $\Psi(t, T)$  satisfies the following ordinary differential equation:*

$$\begin{aligned} \frac{\partial \Psi(t, T)}{\partial t} &= -\frac{1}{2}(a\Psi(t, T) + b)[\bar{\eta}(T-t)'BB'\bar{\eta}(T-t)] \\ \Psi(0, T) &= X_0(T) \end{aligned}$$

**Theorem 2.** *Given four constant parameters  $a, b, c,$  and  $k$  of the static transformation  $\Phi(x)$  and vector  $\bar{\xi}$  of the values at zero, a closed-form solution for the vector of the driver functions  $\bar{\eta}(\tau)$  is given by the formula*

$$\bar{\eta}(\tau) = (1 - a\omega(\tau))^{-\frac{1}{2}} e^{-K'\tau} \bar{\xi}$$

*Where*

$$\omega(\tau) = \int_0^{\tau} \bar{\xi}' e^{-Ks} C e^{-K's} \bar{\xi} ds, \quad C = BB'$$

*Function  $\Psi(t, T)$  is given by the formula*

$$\Psi(t, T) = X_0(T) + \frac{1}{2} b [\omega(T-t) - \omega(T)], \quad \text{if } a = 0$$

$$\Psi(t, T) = \left( X_0(T) + \frac{b}{a} \right) \left( \frac{1 - a\omega(T)}{1 - a\omega(T-t)} \right)^{\frac{1}{2}} - \frac{b}{a}, \quad \text{if } a \neq 0$$

It is well known that the most general form of the matrix exponents in the formula for the driver functions  $\bar{\eta}(\tau)$  above is a product of polynomials, exponential functions and sine/cosine functions. Sine/cosine functions appear only in the case when the matrix  $K$  of mean reversion speeds has complex characteristic values. However, an empirical evidence for the Commodity markets does not support any “periodicity” or long-term “wave” shapes in the futures curve historical volatility term structures as functions of time to maturity  $\tau$  (the absence of “periodicity” for the Natural Gas futures volatility as a function of time to maturity  $\tau$  should not be mistakenly substituted by seasonality in the volatility as a function of maturity  $T$ ). This property makes it possible to suppose only real characteristic values for the mean reversion speed matrix  $K$ . Therefore, general representations of the driver functions and volatility functions involve products of polynomials and exponential functions. A full description of all different types for considered futures curve models is based on a well-known explicit analytical representation of a matrix exponent using Jordan canonical decomposition of a general matrix with real characteristic values.

**Proposition 4.** For any  $M \times M$  matrix  $\mathbf{K}$  with real (non-negative) characteristic values,  $\lambda_i \geq 0$ , the matrix  $-\mathbf{K}'$  can be represented in the following Jordan canonical form:

$$-\mathbf{K}' = \mathbf{G}\mathbf{\Lambda}\mathbf{G}^{-1}, \quad \det \mathbf{G} \neq 0,$$

$$\mathbf{\Lambda} = \text{diag}(\mathbf{\Lambda}_1, \dots, \mathbf{\Lambda}_m), \quad \mathbf{\Lambda}_i = \text{diag}(\mathbf{\Lambda}_{i1}, \dots, \mathbf{\Lambda}_{li}), \quad \mathbf{\Lambda}_{ij} (n_{ij} \times n_{ij}),$$

$$\mathbf{\Lambda}_{ij} = \begin{vmatrix} -\lambda_i & 1 & 0 & \cdot & 0 & 0 & 0 \\ 0 & -\lambda_i & 1 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & -\lambda_i & 1 \\ 0 & 0 & 0 & \cdot & 0 & 0 & -\lambda_i \end{vmatrix}, \quad \lambda_i \geq 0$$

The matrix exponent  $e^{-\mathbf{K}'\tau}$  that is a fundamental matrix for the system of linear ODEs  $\bar{\eta}'(\tau) = -\mathbf{K}'\bar{\eta}(\tau)$  is represented by the formulas:

$$e^{-\mathbf{K}'\tau} = \mathbf{G}e^{\mathbf{\Lambda}\tau}\mathbf{G}^{-1}, \quad e^{\mathbf{\Lambda}\tau} = \text{diag}(e^{\mathbf{\Lambda}_{11}\tau}, \dots, e^{\mathbf{\Lambda}_{l_1 l_1}\tau}, \dots, e^{\mathbf{\Lambda}_{m l_m l_m}\tau}), \quad e^{\mathbf{\Lambda}_{ij}\tau} = e^{-\lambda_i \tau} \begin{vmatrix} 1 & t & \frac{t^2}{2!} & \cdot & \frac{t^{n_{ij}-1}}{(n_{ij}-1)!} \\ 0 & 1 & t & \cdot & \frac{t^{n_{ij}-2}}{(n_{ij}-2)!} \\ 0 & 0 & 1 & \cdot & \frac{t^{n_{ij}-3}}{(n_{ij}-3)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 \end{vmatrix}$$

If all characteristic values  $\lambda_i > 0$  of the matrix  $K$  are single then the matrix exponent is diagonal with the exponential functions  $\exp(-\lambda_i \tau)$  on the diagonal resulting in well-known exponential volatility functions. Polynomial blocks (multiplied by the same exponential function) may appear only for two-diagonal Jordan blocks of size greater or equal two. This case of more general volatility functions (for linear and exponential models with parameter  $a = 0$  in the static transformation) still corresponds to mean-reverting dynamics of the futures curves. If some characteristic value  $\lambda_i$  of the mean reversion matrix  $K$  is zero (i.e. the stochastic system has a non-mean reverting Wiener component) then the multiplicity of this  $\lambda_i$  should be one. If we suppose the contrary, then the driver functions and the corresponding volatility functions would be polynomials of the order higher than zero. This means that the volatility functions grow infinitely for long time to maturity. Such volatility functions are not observed in the market. Therefore, for linear and exponential models ( $a = 0$ ) widely accepted in Commodity Industry, the driver and volatility functions for the co-integrated case with non-zero Wiener component can be linear combinations of a constant volatility function and exponentially decreasing functions multiplied by polynomials. Consequently, in the case of pure mean reversion models from the considered family, the volatility of the futures curves goes to zero for  $\tau \rightarrow +\infty$ . In the case of co-integrated models, the volatility has a horizontal positive asymptote. The latter case is observed for Crude Oil and Natural Gas futures markets.

**Linear Model.** For the parameters  $a = 0$ ,  $b = 0$ ,  $\tilde{c} \neq 0$ , and arbitrary  $k$  the static transformation is linear  $\Phi(x) = \tilde{c}x + k$  (therefore, the Commodity futures and spot prices have Gaussian distributions, and they can be negative). The driver functions  $\bar{\eta}(\tau)$  satisfy the system of linear ODEs  $\bar{\eta}'(\tau) = -K'\bar{\eta}(\tau)$ . They are represented by the formula  $\bar{\eta}(\tau) = e^{-K'\tau} \bar{\xi} = Ge^{\Lambda\tau} G^{-1} \bar{\xi}$ . In one-dimensional case for  $\lambda_1 = 0$ , this model is a Bachelier 1900 model with a flat volatility and Wiener driver. If all eigenvalues  $\lambda_j > 0$  are single then the model is a multivariate analog of the mean reversion Vasicek model with exponential volatilities. If there are multiple eigenvalues  $\lambda_j > 0$  then the model is an “extended” Vasicek-type mean reversion model with the new volatility functions equal to products of polynomials and exponential functions. If one of the eigenvalues is zero then the model is a co-integrated “extended Bachelier-Vasicek – type” model.

**Exponential Model.** For  $a = 0$ ,  $b \neq 0$ ,  $\tilde{c} \neq 0$  and any  $k$ , the static transformation is exponential  $\Phi(x) = \tilde{c} e^{bx} + k$  (therefore, the Commodity futures prices and spot prices have log-normal distributions, and they are positive). The driver functions  $\bar{\eta}(\tau)$  satisfy the same system of linear ODEs as those for the Linear Model. Therefore, the volatility functions are the same. In one-dimensional case for  $\lambda_1 = 0$ , this model is the famous Black's model. If all eigenvalues  $\lambda_j > 0$  are single then the model is a multivariate analog of the mean reversion Schwartz model with exponential volatilities (similar to the extended Black - Karasinski model in the interest rate theory). If there are multiple eigenvalues  $\lambda_j > 0$  then the model is an "extended" exponential mean reversion model with new volatility functions equal to products of polynomials and exponential functions. If one of the eigenvalues is zero then the model is a co-integrated generalization of the "two-factor Schwartz" model with the flat volatility function for the Wiener component in the log-prices and exponential (exponential – polynomial) volatility functions for the mean reversion component in the log-prices.

**Capped Futures Price Model<sup>1</sup>.** For the parameters  $a < 0$ ,  $\tilde{c} \neq 0$  and arbitrary  $b$  and  $k$ , the static transformation  $\Phi(x)$  is a Normal Cumulative Distribution Function (NCDF) multiplied and shifted up by some constant parameters  $L$  and  $k$  (we assume  $L > 0$  and  $k \geq 0$ ):

$$\Phi(x) = \tilde{c} \int_{-\infty}^x e^{\frac{1}{2}as^2 + bs} ds + k = L N(x; m_{\Phi}, \sigma_{\Phi}) + k, \quad N(x; m_{\Phi}, \sigma_{\Phi}) = \frac{1}{\sqrt{2\pi}\sigma_{\Phi}} \int_{-\infty}^x e^{-\frac{(s-m_{\Phi})^2}{2\sigma_{\Phi}^2}} ds$$

Therefore, the commodity (interest rate or other underlying) futures prices  $F(t, T)$  and spot price (short interest rate)  $S(t) = F(t, t)$  are capped (bounded):

$k < F(t, T) < m$ ,  $m = k + L$ . The driver functions and function  $\Psi(t, T)$  are presented by the general formulas in Theorem 2. The model has a closed analytical form for the risk-neutral density and non-Black-Scholes formulas for European options on futures (in the form of an integral).

**“Explosive” Futures Price Model.** For the parameters  $a > 0$ ,  $\tilde{c} \neq 0$  and arbitrary  $b$  and  $k$ , the static transformation  $\Phi(x)$  is a Dowson’s Integral-type function:

$$\Phi(x) = \int_0^x e^{\frac{1}{2}as^2 + bs + c} ds + k$$

Description of this model is included in the full article.

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<sup>1</sup> As of our knowledge, this is the first no-arbitrage futures model with capped futures prices. It may be useful for some Electricity and other commodity markets, but also for the interest rate futures HJM-type methodology which has problems for the models with a log-normal short interest rate resulting in infinite mathematical expectation for the bank account for any short period of time.

One can show that for the constant prices of risk associated with the driver Brownian motions the considered class of the futures curve models remains the same in the real world measure as the one in the risk neutral measure, but the driver Gaussian system has non-zero means (or drifts):

$$d\bar{z}(t) = \mathbf{K}(\bar{\mu} - \bar{z}(t))dt + B d\bar{w}$$

Considered class of diffusion models can be generalized to the state variables driven by Lévy processes:

$$F(t, T) = \Phi(X(t, T)), \quad \Phi'(x) \neq 0$$

$$X(t, T) = \sum_{j=1}^M \eta_j (T - t) z_j(t) + \Psi(t, T) = \bar{\eta}(T - t)' \bar{z}(t) + \Psi(t, T)$$

$$d\bar{z}(t) = -\mathbf{K} \bar{z}(t) dt + B d\bar{w} + d\bar{L}, \quad \bar{z}(t) = (z_1(t), \dots, z_M(t))', \quad M \geq 1$$

$$\bar{z}(0) = \bar{0}$$

$$X(0, T) = \Psi(0, T) = \Phi^{-1}(F(0, T)) = X_0(T)$$

For example, in one-dimensional case, assuming that the Lévy measure,  $\nu(dy)$ , of the Lévy process  $L$  satisfies the condition  $\int_{|y|\leq 1} |y| \nu(dy) < \infty$ , one can derive from general results of [Bjork, Landen, 2000] the integro-differential equation of no-arbitrage condition for this class of models:

$$\left\{ \frac{\partial \Psi}{\partial t} \eta + [-\Psi + x][K\eta - \eta'] \right\} \Phi'(x) + \frac{1}{2} B^2 \eta^3 \Phi''(x) + \eta \int_{-\infty}^{+\infty} [\Phi(x + y\eta) - \Phi(x)] \nu(dy) \equiv 0$$

It should be valid for all  $x$ . For linear and exponential transformations  $\Phi(x)$ , this condition leads to exponential driver functions  $\eta(\tau)$  (as for pure diffusion case) and close-form solution for  $\Psi(t, \tau)$ . Existence of other non-linear transformations and their analytical form is still an open problem.