White Noise Approach to Interest Rate Models

Alexei Filinkov and John van der Hoek
Adelaide University, SA 5005

Abstract

We discuss the class of models for the term structure of forward interest rates when the dynamics involve the following stochastic evolution equation:

$$dX(t) = \big[AX(t) + F(t)\big]dt + BdW(t),\ X(0) = X_0,$$

where X takes values in a separable Hilbert space H. Here A is the generator of a semigroup, $B : H \to H$ is a bounded linear operator, $W(\cdot)$ is an H-valued cylindrical Wiener process. This model includes in particular the HJM model (in the parametrization of Musiela), second order term structure models and generalizations.

Mathematics Subject Classification 2000: 91B28, 60H40, 60H30, 60G20, 47D06, 60J70

1 Introduction

We study models for the term structure of forward interest rates when the dynamics involve the following stochastic evolution equation:

$$dX(t) = \Big[AX(t) + F(t)\Big]dt + BdW(t),\ X(0) = X_0, \tag{1}$$

where X takes values in a separable Hilbert space H. Here A is the generator of a semigroup, $B : H \to H$ is a bounded linear operator, $W(\cdot)$ is an (infinite dimensional) H-valued cylindrical Wiener process.

These models arise both when modelling forward interest rates in the real world sense (for use in econometrics and risk analysis) and in the risk neutral formulation (for use in derivatives pricing). We refer the reader to [3], [4], [7] for further discussion and note that the questions of consistency of such models are discussed in [19]. The motivation for using the infinite dimensional noise in these models can be summarized as follows:

(1) the non arbitrage conditions are too restrictive on drift when only finite number of Brownian motions is used in modelling the noise;

(2) it is more natural to solve a model with an infinite dimensional noise and then to select for appropriate purposes a model with a finite dimensional noise.

We focus on the following three examples.

(a) HJM Model in Musiela Parametrization

Let $X(t,\xi)$ be the forward rate function, where $t \geq 0$ is time and $\xi \in [0,M]$ is time to maturity. The Gaussian version of the Heath-Jarrow-Morton (HJM) model in Musiela parametrization under the assumption of absence of an arbitrage strategy is described in risk neutral form by the following equation [2]:

$$\begin{aligned} dX(t,\xi) &= \left[\frac{\partial X}{\partial \xi}(t,\xi) + \sum_{j=1}^{d} \sigma_j(t,\xi) \int_0^\xi \sigma_j(t,\eta) d\eta\right] dt + \sum_{j=1}^{d} \sigma_j(t,\xi) d\beta_j(t), \quad t \geq 0,\, \xi \in [0,M], \\ X(0,\xi) &= X_0(\xi), \end{aligned}$$

where β_j, $j = 1,\dots,d$, are independent Brownian motions. In the case of unbounded maturities $M = +\infty$ and $X(t)$ has values in the Hilbert space $H = L^2_\gamma(0,\infty)$ of measurable functions with

$$\|x\|_\gamma^2 = \int_0^\infty e^{-\gamma\xi}|x|^2 d\xi < \infty.$$

In the case of bounded maturities $H = L^2(0,M)$ and $X(t,M) = l(t),\ t \geq 0$.

As in [10] we introduce the operator $\mathcal{S}$ on H by

$$(\mathcal{S}f)(x) := f(x) \int_0^x f(\eta) d\eta, \quad x \geq 0.$$

The following generalization of the HJM model to the case of infinite dimensional noise was studied in [10]:

$$\begin{aligned} dX(t,\xi) &= \left[\frac{\partial X}{\partial \xi}(t,\xi) + \sum_{j=1}^{\infty} \mathcal{S}\sigma_j(t,\xi)\right] dt + \sigma(t,\xi) dW(t), \quad t,\xi \geq 0, \\ X(0,\xi) &= X_0(\xi)\,, \end{aligned}$$

where $\sigma = \sum_{j=1}^\infty \sigma_j$.

(b) LIBOR Model

Let $L(t,\xi)$ be the LIBOR rate process: see [2], [8]. Then the process $X(t,\xi) = \log L(t,\xi)$ satisfies the following equation on $H = L^2_\gamma(0,\infty)$:

$$dX(t) = \Big[AX(t) + F(t,X)\Big] dt + \Gamma(t) dW(t),\ X(0) = X_0\,,$$

for some Γ and F.

(c) Second Order Term Structure Models

Let again $X(t,\xi)$ be the forward rate function. Then $s(t) = X(t,0),\ t \geq 0$, is the short rate and $l(t) = X(t,M),\ t \geq 0$, is the long rate. Let $m(\xi)$ be the average profile of the term structure. Note that $m(\cdot)$ is a deterministic function and also without loss of generality one can assume that $m(0) = 0$ and $m(M) = 1$. Then the term structure can be represented in the form:

$$X(t,\xi) = s(t) + \Big(l(t) - s(t)\Big)\Big[m(\xi) + Y(t,\xi)\Big]\,, \qquad t \geq 0,\, \xi \in [0,M],$$

where $Y(t,\xi),\ t \geq 0,\ \xi \in [0,M]$, is the fluctuation (or deformation) process. Note that $Y(t,0) = Y(t,M) = 0$ as $m(0) = 0$ and $m(M) = 1$. As suggested in [4], it is natural to assume that the fluctuation process $Y(t) \equiv Y(t,\xi),\ t \geq 0$, has values in the separable Hilbert space $H = L^2(0,M)$, and its dynamics is described by the following stochastic differential equation

$$\begin{aligned} dY(t,\xi) &= \left[\frac{\partial Y}{\partial \xi}(t,\xi) + b(t,\xi,Y) + \frac{\kappa}{2}\frac{\partial^2 Y}{\partial \xi^2}(t,\xi)\right]dt + \sigma(t,\xi,Y)dW(t), \quad t \geq 0,\ \xi \in [0,M], \\ Y(0,\xi) &= Y_0(\xi). \end{aligned}$$

Here $W(\cdot)$ is an (infinite dimensional) H-valued cylindrical Wiener process, b and σ are some drift and volatility functions, and $\kappa > 0$. The derivatives $\frac{\partial Y}{\partial \xi}$ and $\frac{\partial^2 Y}{\partial \xi^2}$ represent the steepness of the term structure and its curvature, respectively. This is a ‘real world’ formulation. Its risk neutral formulation was left as an open question by R. Cont [4]. It is also assumed here that process $Y(\cdot)$ is independent from the short and long rate processes $s(\cdot)$ and $l(\cdot)$, and the process $(s(\cdot), l(\cdot))$ can be modelled as a bivariate diffusion process [4].

In this paper we use the white noise approach for treating equation (1). White noise analysis was developed extensively in the past three decades, see for example [12], [13], [15], [14] and references therein. In the recent paper [9] we developed the H-valued white noise analysis (here H is a separable Hilbert space), using the approach of [14], where the theory of $\mathbb{R}^N$-valued stochastic distributions is presented.

2 Preliminaries

In this section we give a very brief summary of the abstract white noise calculus that was developed in [9].

Consider the probability space $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)), \mu)$ where $S'(\mathbb{R}^d)$ is the space of tempered distributions, and μ is the unique probablility measure on $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d))$ satisfying

$$\int_{S'(\mathbb{R}^d)} e^{i\langle \omega, \phi\rangle} d\mu(\omega) = e^{-1/2\|\phi\|^2_{L^2(\mathbb{R}^d)}} \ ,$$

where $\langle \omega, \phi\rangle$ denotes the action of $\omega \in S'(\mathbb{R}^d)$ on $\phi \in S(\mathbb{R}^d)$. We denote the space $L^2(S'(\mathbb{R}^d), \mu)$ of square integrable functions on $S'(\mathbb{R}^d)$ with values in $\mathbb{R}$ by $L^2(\mu)$. Let H be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^\infty$, and $L^2(H)$ denote the space $L^2(S'(\mathbb{R}^d), \mu; H)$ of square integrable functions on $S'(\mathbb{R}^d)$ with values in H. The inner product in $L^2(H)$ is defined by

$$\langle f, g\rangle_{L^2(H)} := \int_{S'(\mathbb{R}^d)} \langle f, g\rangle_H \, d\mu.$$

We will use the classical Wiener-Itô chaos expansion of elements of $L^2(\mu)$ in terms of Hermite polynomials (see, for example, [14]):

$$h_n(x) = (-1)^n e^{1/2x^2} \frac{d^n}{dx^n}(e^{-1/2x^2}), \quad n = 0, 1, 2, \dots \ .$$

The Hermite functions are defined by

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-1/2x^2} h_{n-1}(x) \ , \quad n = 1, 2, \dots \ ,$$

are used to define the following orthonormal basis for $L^2(\mathbb{R}^d)$, the family of tensor products

$$\eta_i := \xi_{\delta_1^{(i)}} \otimes \ldots \otimes \xi_{\delta_d^{(i)}}, \quad i = 1, 2, \ldots ,$$

where $\delta_1^{(i)}, \ldots, \delta_d^{(i)} \in \mathbb{N}$ are chosen so that if $i < j$

$$\delta_1^{(i)} + \delta_2^{(i)} + \ldots + \delta_d^{(i)} \leq \delta_1^{(j)} + \delta_2^{(j)} + \ldots + \delta_d^{(j)} .$$

The orthogonal basis for $L^2(\mu)$ is constructed as the family of functions $\{H_\alpha\}_{\alpha \in \mathcal{J}}$ defined by

$$H_\alpha(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle), \quad \omega \in S'(\mathbb{R}^d) \ ,$$

where $\mathcal{J} = (\mathbb{N}_0^{\mathbb{N}})_c$ is the space of sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\alpha_1, \alpha_2, \ldots \in \mathbb{N}_0$, such that there are only finitely many $\alpha_i \neq 0$. The $L^2(\mu)$-norm of these $\{H_\alpha\}_{\alpha \in \mathcal{J}}$ is

$$\|H_\alpha\|^2_{L^2(\mu)} = \alpha! := \alpha_1! \alpha_2! \ldots .$$

Any $f \in L^2(\mu)$ has the following representation

$$f = \lim_{k \to \infty} \sum_{\text{index } \alpha \leq k} c_\alpha H_\alpha = \lim_{n \to \infty} \sum_{\alpha \in \Gamma_n} c_\alpha H_\alpha \, ,$$

where $c_\alpha = (\alpha!)^{-1} \langle f, H_\alpha \rangle_{L^2(\mu)}$, and we use the notation:

$$\begin{aligned} &\text{index } \alpha := \sup\{k \mid \alpha_k \neq 0\} \\ &\Gamma_n := \{\alpha \in \mathcal{J} \mid \alpha_i \leq n, \ \forall i \leq n \text{ and } \alpha_i = 0, \ \forall i > n\} \, . \end{aligned}$$

We also denote $\lim_{k \to \infty} \sum_{\text{index } \alpha \leq k}$ and $\lim_{n \to \infty} \sum_{\alpha \in \Gamma_n}$ by $\sum_{\alpha \in \mathcal{J}}$.

An orthogonal basis for $L^2(H)$ is constructed in the following lemmas.

Lemma 1 *For a random variable $f \in L^2(H)$, there exists a sequence of random variables $a_i \in L^2(\mu)$ so that the sum*

$$f = \sum_{i=1}^{\infty} a_i e_i \ ,$$

converges in $L^2(H)$.

Lemma 2 *The family of functions $\{H_\alpha(\omega) e_i\}_{i \in \mathbb{N}, \alpha \in \mathcal{J}}$ is an orthogonal basis for $L^2(H)$.*

We now define the spaces $S(H)_\rho$ of H-valued stochastic test functions and the spaces $S(H)_{-\rho}$ of H-valued stochastic distributions.

Definition 1 *1. For $\rho \in [0, 1]$, define $S(H)_\rho$ to consist of all*

$$f(\omega) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i\alpha} H_\alpha(\omega) e_i \, , \quad c_{i\alpha} \in \mathbb{R} \, ,$$

in $L^2(H)$ such that for all $k \in \mathbb{N}$

$$\|f\|^2_{\rho,k} \ := \ \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} (\alpha!)^{1+\rho} c_{i\alpha}^2 (2\mathbb{N})^{k\alpha} = \sum_{i=1}^{\infty} \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1+\rho} c_{i\alpha}^2 (2\mathbb{N})^{k\alpha} < \infty \, ,$$

where $(2\mathbb{N})^{k\alpha} := \prod_{j=1}^{\infty} (2j)^{k\alpha_j}$.

2. *For $\rho \in [0,1]$, define $S(H)_{-\rho}$ to consist of all formal expansions*

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i\alpha} H_\alpha(\omega) e_i \ , \quad c_{i\alpha} \in \mathbb{R} \ ,$$

   such that for some $q \in \mathbb{N}$

$$\|F\|^2_{-\rho,-q} \ := \ \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} (\alpha!)^{1-\rho} c^2_{i\alpha} (2\mathbb{N})^{-q\alpha} = \sum_{i=1}^{\infty} \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} c^2_{i\alpha} (2\mathbb{N})^{-q\alpha} < \infty \ .$$

Note that for $\rho \in [0,1]$

$$S(H)_1 \subset S(H)_\rho \subset S(H)_0 \subset L^2(H) \subset S(H)_{-0} \subset S(H)_{-\rho} \subset S(H)_{-1} \ .$$

For $\rho \in [0,1]$ and $f(\omega)$ and $F(\omega)$ belonging to $S(H)_\rho$ and $S(H)_{-\rho}$ respectively, we can write

$$f(\omega) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i\alpha} H_\alpha(\omega) e_i = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega) = \sum_{i=1}^{\infty} f_i(\omega) e_i \ , \quad f_i(\omega) \in (S)_\rho \ ,$$

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_{i\alpha} H_\alpha(\omega) e_i = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega) = \sum_{i=1}^{\infty} F_i(\omega) e_i \ , \quad F_i(\omega) \in (S)_{-\rho} \ ,$$

where $c_\alpha = \sum_{i=1}^{\infty} c_{i\alpha} e_i \in H$. For $k, q \in \mathbb{N}$

$$\|f\|^2_{\rho,k} = \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1+\rho} \|c_\alpha\|^2_H (2\mathbb{N})^{k\alpha} = \sum_{j=1}^{\infty} |f_j|^2_{\rho,k} \ ,$$

$$\|F\|^2_{-\rho,-q} = \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} \|c_\alpha\|^2_H (2\mathbb{N})^{-q\alpha} = \sum_{j=1}^{\infty} |F_j|^2_{-\rho,-q} \ .$$

In order to describe the topology of $S(H)_\rho$ and $S(H)_{-\rho}$ we consider the following spaces.

Definition 2 1. *For $\rho \in [0,1]$ and $k \in \mathbb{N}$, define $S(H)_{\rho,k}$ to consist of all*

$$f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha, \quad c_\alpha \in H \, ,$$

   in $L^2(H)$ such that $\|f\|_{\rho,k} < \infty$.

   Define the inner product of two elements

$$f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \ , \quad g = \sum_{\alpha \in \mathcal{J}} d_\alpha H_\alpha \, ,$$

   belonging to $S(H)_{\rho,k}$ by

$$\langle f, g \rangle_{\rho,k} := \sum_{\alpha \in \mathcal{J}} \langle c_\alpha, d_\alpha \rangle_H (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} \, .$$

2. *For $\rho \in [0,1]$ and $q \in \mathbb{N}$, define $S(H)_{-\rho,-q}$ to consist of all formal expansions*

$$F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha, \quad c_\alpha \in H \, ,$$

   such that $\|F\|_{-\rho,-q} < \infty$.

We can see that for $\rho \in [0,1]$

$$S(H)_\rho = \cap_{k=1}^{\infty} S(H)_{\rho,k} \ , \quad S(H)_{-\rho} = \cup_{q=1}^{\infty} S(H)_{-\rho,-q} \ .$$

The following results are proved in [9].

Proposition 1 *For $\rho \in [0,1]$ and $k \in \mathbb{N}$, $S(H)_{\rho,k}$ equipped with $\langle \cdot,\cdot \rangle_{\rho,k}$ is a separable Hilbert space.*

Proposition 2 *For $\rho \in [0,1]$, the system of inner products $\langle \cdot,\cdot \rangle_{\rho,k}$ on $S(H)_\rho$ is compatible.*

Corollary 1 *For $\rho \in [0,1]$, $S(H)_\rho$ equipped with the countable collection of inner products $\langle \cdot,\cdot \rangle_{\rho,k}$ is countably Hilbert.*

Proposition 3 *For $\rho \in [0,1]$ and $k \in \mathbb{N}$, $S(H)_{-\rho,-k}$ is the dual of $S(H)_{\rho,k}$.*

Corollary 2 *For $\rho \in [0,1]$, $S(H)_{-\rho}$ is the dual of $S(H)_\rho$.*

Proposition 4 *Consider $F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \in S(H)_{-\rho}$ and $f = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in S(H)_\rho$, $\rho \in [0,1]$. The action of F on f is*

$$\langle F, f \rangle = \sum_{\alpha \in \mathcal{J}} \alpha! \langle a_\alpha, c_\alpha \rangle_H \ .$$

Proposition 5 *If $F_n \to F$ strongly in $S(H)_{-1}$, then there exists a $q \in \mathbb{N}$ so that F_n, F belong to $S(H)_{-1,-q}$ and $\|F_n - F\|_{-1,-q} \to 0$.*

Examples

Consider $\{\beta_i(t)\},\ i \in \mathbb{N}, t \geq 0$, defined by

$$\beta_i(t)(\omega) = \sum_{j=1}^{\infty} \int_0^t \xi_j(s) ds \ H_{\varepsilon_{n(i,j)}}(\omega) \ , \tag{2}$$

where $\varepsilon_n = (0, \cdots, 1, 0 \cdots)$ with 1 on the n-th place, 0 otherwise and $n(i,j)$ are chosen according to the following table

j i	1	2	3	4	5	6	7	$\cdots$
1	1	3	6	10	15	21	28	$\cdots$
2	2	5	9	14	20	27		
3	4	8	13	19	26			
4	7	12	18	25				
5	11	17	24					$n(i,j)$
6	16	23						
7	22							
	$\cdots$							

This defines $\{\beta_i(t)\},\ i \in \mathbb{N},\ t \geq 0$, a sequence of independent Brownian motions.

Note that different β_i-s have disjoint families of $H_{\varepsilon_{n(i,j)}}$ in their representations. Now we rewrite (2) as

$$\beta_i(t) = \sum_{k=1}^{\infty} \theta_{ik}(t) H_{\varepsilon_k} \,, \tag{3}$$

where

$$\theta_{ik}(t) = \begin{cases} \int_0^t \xi_j(s)ds & , \ k = n(i,j) \\ 0 & , \ \text{otherwise} \end{cases} .$$

Example 1 (H-valued cylindrical Wiener process)
If $\{\beta_i(\cdot)\}_{i=1}^{\infty}$ is a sequence of independent $\mathbb{R}$-valued Brownian motions, then the following formal sum

$$W(t) := \sum_{i=1}^{\infty} \beta_i(t) e_i, \quad t \geq 0$$

is called an *H-valued Wiener process.* We can rewrite $W(t)$ in the form

$$\begin{aligned} W(t) &= \sum_{i=1}^{\infty} \beta_i(t) e_i = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \theta_{ik}(t) H_{\varepsilon_k} e_i = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \theta_{ik}(t) e_i \right) H_{\varepsilon_k} \\ &= \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left(\int_0^t \xi_j(s)ds \ e_i \right) H_{\varepsilon_k} =: \sum_{k=1}^{\infty} \theta_k(t) H_{\varepsilon_k} \ , \quad \theta_k(t) = \delta_{n(i,j),k} \int_0^t \xi_j(s)ds \ e_i \ . \end{aligned}$$

For all $t \in [0, \infty)$, this sum does not convergent in $L^2(H)$, but does in $S(H)_{-0}$, as for $q > 1$

$$\sum_{k=1}^{\infty} (\varepsilon_k!) \|\theta_k(t)\|_H^2 (2k)^{-q} \ \leq \ \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left(\int_0^t \xi_j(s)^2 ds \right) (2k)^{-q} \leq \sum_{k=1}^{\infty} (2k)^{-q} < \infty \ .$$

Example 2 (H-valued singular white noise process)
The following formal sum

$$\mathbb{W}(t) := \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left(\xi_j(t) e_i \right) H_{\varepsilon_k} = \sum_{k=1}^{\infty} \kappa_k(t) H_{\varepsilon_k}, \quad t \geq 0, \quad \kappa_k(t) = \delta_{n(i,j),k} \xi_j(t) e_i \ ,$$

is called an *H-valued singular white noise process.* Similarly to the previous example we obtain that $\mathbb{W}(t) \in S(H)_{-0-q}$, $q > 1$, for each $t \geq 0$.

Example 3 (Q-Wiener process)
Let Q be a positive, trace class (i.e. finite trace) operator on H with eigenvalues $\lambda_i > 0$. We call the following sum

$$\begin{aligned} W_Q(t) &:= \sum_{i=1}^{\infty} \sqrt{\lambda_i} \beta_i(t) e_i = \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left(\sqrt{\lambda_i} \int_0^t \xi_j(s)ds \ e_i \right) H_{\varepsilon_k} \\ &= \sum_{k=1}^{\infty} \vartheta_k(t) H_{\varepsilon_k}, \quad t \geq 0, \quad \vartheta_k(t) = \delta_{n(i,j),k} \sqrt{\lambda_i} \int_0^t \xi_j(s)ds \ e_i, \end{aligned}$$

a *Q-Wiener process.* For each $t \geq 0$, $W_Q(t)$ belongs to $L^2(H)$ as

$$\begin{aligned} \|W_Q(t)\|_{L^2(H)}^2 &= \sum_{k=1}^{\infty} (\varepsilon_k!) \|\vartheta_k\|_H^2 = \sum_{k=1}^{\infty} \delta_{n(i,j),k} \left(\sqrt{\lambda_i} \int_0^t \xi_j(s)ds \right)^2 \\ &\leq \sum_{i=1}^{\infty} \lambda_i \sum_{j=1}^{\infty} \left(\int_{\mathbb{R}} I_{[0,t]}(s) \xi_j(s)ds \right)^2 = \|I_{[0,t]}\|_{L^2(\mathbb{R})}^2 \sum_{i=1}^{\infty} \lambda_i < \infty \,. \end{aligned}$$

Wick Product

Consider $F, G \in S(H)_{-1}$, having forms

$$F = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty} c_{i\alpha}H_\alpha e_i = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha = \sum_{i=1}^{\infty} F_i e_i \ , \ c_\alpha \in H, \ F_i \in (S)_{-1} \ ,$$

$$G = \sum_{\beta\in\mathcal{J}}\sum_{i=1}^{\infty} d_{i\beta}H_\beta e_i = \sum_{\beta\in\mathcal{J}} d_\beta H_\beta = \sum_{i=1}^{\infty} G_i e_i \ , \ d_\beta \in H, \ G_i \in (S)_{-1} \ .$$

Definition 3 *The Wick product of elements* $F, G \in S(H)_{-1}$ *is*

$$(F \diamond G)(\omega) \ := \ \sum_{i=1}^{\infty}(F_i \diamond G_i)(\omega)e_i = \sum_{\gamma\in\mathcal{J}}\sum_{i=1}^{\infty}\left(\sum_{\alpha+\beta=\gamma} c_{i\alpha}d_{i\beta}\right)H_\gamma(\omega)e_i = \sum_{\gamma\in\mathcal{J}} g_\gamma H_\gamma(\omega)\,,$$

where

$$g_\gamma = \sum_{i=1}^{\infty}\sum_{\alpha+\beta=\gamma} c_{i\alpha}d_{i\beta}e_i =: \sum_{i=1}^{\infty} g_{i\gamma}e_i\,.$$

It is not difficult to show that the space $S(H)_{-1}$ is invariant under Wick multiplication.

Proposition 6 *If* $F, G \in S(H)_{-1}$, *then* $F \diamond G \in S(H)_{-1}$.

Proposition 7 *If* $F \in S(H)_{-0}$, *then* $F \diamond \mathbb{W}(t) \in S(H)_{-0}$.

Consider $F \in (S)_{-1}$ and $G \in S(H)_{-1}$, having forms

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha \in (S)_{-1} \ , \ c_\alpha \in \mathbb{R} \ , \ G = \sum_{\beta\in\mathcal{J}}\sum_{i=1}^{\infty} d_{i\beta}H_\beta e_i \in S(H)_{-1}\,.$$

Definition 4 *The Wick product of elements* $F \in (S)_{-1}$ *and* $G \in S(H)_{-1}$ *is defined by*

$$(F \diamond G)(\omega) \ := \ \sum_{i=1}^{\infty}\left(\sum_{\alpha+\beta=\gamma} c_\alpha d_{i\beta}\right)H_\gamma(\omega)e_i = \sum_{\gamma\in\mathcal{J}} g_\gamma H_\gamma(\omega)\,,$$

where

$$g_\gamma = \sum_{i=1}^{\infty}\sum_{\alpha+\beta=\gamma} c_\alpha d_{i\beta}e_i\,.$$

One can easily prove the following properties.

Proposition 8 **(a)** *If* $F \in (S)_{-1}, G \in S(H)_{-1}$, *then* $F \diamond G \in S(H)_{-1}$.

(b) *If* $F \in (S)_{-0}$ *then* $F \diamond \mathbb{W}(t) \in S(H)_{-0}$.

Example 4 Let F be deterministic, that is $F = c_0 = \sum_{i=1}^{\infty} F_i e_i$, $F_i \in \mathbb{R}$, and let $G \in S(H)_{-1}$. Then

$$(F \diamond G)(\omega) \ = \ \sum_{i=1}^{\infty}(F_i G_i)\,e_i = \sum_{i=1}^{\infty}\left(\sum_{\beta\in\mathcal{J}} F_i d_{i\beta}H_\beta(\omega)\right)e_i = \sum_{\beta\in\mathcal{J}} g_\beta H_\beta(\omega)\ .$$

Hitsuda-Skorohod Integration

Definition 5 *We say that a process $F(t) : \mathbb{R} \to S(H)_{-0}$ is Pettis integrable if*

$$\langle F(t), \phi \rangle \in L^1(\mathbb{R}, dt) \text{ for all } \phi \in S(H)_0 \ .$$

In this case the Pettis integral of F is the unique element of $S(H)_{-0}$ defined by

$$\langle \int_{\mathbb{R}} F(t)dt, \phi \rangle = \int_{\mathbb{R}} \langle F(t), \phi \rangle dt, \quad \phi \in S(H)_0 \ .$$

The existence of the unique element $\int_{\mathbb{R}} F(t)dt \in S(H)_{-0}$ follows from the fact that

$$\int_{\mathbb{R}} \langle F(t), \cdot \rangle dt$$

is a continuous linear operator on $S(H)_0$.

Lemma 3 *Consider $F(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) H_\alpha(\omega) \in S(H)_{-0}$, $c_\alpha(t) \in H$, where*

$$\sum_{\alpha \in \mathcal{J}} \alpha! \left(\int_{\mathbb{R}} \|c_\alpha(t)\|_H dt \right)^2 (2\mathbb{N})^{-q\alpha} < \infty \ ,$$

for some $q \in \mathbb{N}$. Then $F(t)$ is Pettis integrable and

$$\int_{\mathbb{R}} F(t)dt = \sum_{\alpha \in \mathcal{J}} \left(\int_{\mathbb{R}} c_\alpha(t)dt \right) H_\alpha(\omega) \ .$$

Definition 6 *Suppose $F(t) : \mathbb{R} \to S(H)_{-0}$ (or $(S)_{-0}$) is such that $F(t) \diamond \mathbb{W}(t)$ is Pettis integrable. Then we define the abstract Hitsuda-Skorohod integral of F by*

$$\int_{\mathbb{R}} F(t) \delta W(t) := \int_{\mathbb{R}} F(t) \diamond \mathbb{W}(t) dt \in S(H)_{-0} \ .$$

Proposition 9 *Consider $F(t) : \mathbb{R} \to S(H)_{-0}$ with form*

$$F(t) = \sum_{\alpha \in \mathcal{J}} c_\alpha(t) H_\alpha \in S(H)_{-0} \ .$$

If

$$\sup_{\alpha \in \mathcal{J}} \left\{ \alpha! (2\mathbb{N})^{-q\alpha} \int_{\mathbb{R}} \|c_\alpha(t)\|_H^2 dt \right\} < \infty \ ,$$

for some $q \in \mathbb{N}$, then F is Hitsuda-Skorohod integrable.

Example 5 We have

$$\int_0^t 1 \delta W(\tau) \ = \ \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \left(\int_0^t \kappa_{ik}(\tau) d\tau \right) H_{\varepsilon_k} e_i = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \theta_{ik}(t) H_{\varepsilon_k} e_i = W(t) \ ,$$

where $1 = H_{(0,\ldots)}$.

In general, if $F(\cdot)$ is a deterministic H-valued function satisfying the assumption in proposition 9, then

$$\begin{aligned}
&\int_0^t F(\tau)\delta W(\tau) \\
= \;& \int_0^t F(\tau)\diamond \mathbb{W}(\tau)d\tau = \sum_{k=1}^{\infty}\left(\int_0^t \sum_{i=1}^{\infty} F_i(\tau)\kappa_{ik}(\tau)e_i d\tau\right) H_{\varepsilon_k} \\
= \;& \sum_{j=1}^{\infty}\left(\int_0^t F_1(\tau)\xi_j(\tau)e_1 d\tau\right) H_{\varepsilon_{n(1,j)}} + \sum_{j=1}^{\infty}\left(\int_0^t F_2(\tau)\xi_j(\tau)e_2 d\tau\right) H_{\varepsilon_{n(2,j)}} + \cdots \\
=: \;& \int_0^t F_1(\tau)e_1\delta\beta_1(\tau) + \int_0^t F_2(\tau)e_2\delta\beta_2(\tau) + \cdots = \sum_{i=1}^{\infty}\int_0^t F_i(\tau)e_i\delta\beta_i(\tau)\ ,
\end{aligned}$$

where we naturally introduced the integrals with respect to Brownian motions:

$$\int_0^t F_i(\tau)e_i\delta\beta_i(\tau) := \sum_{j=1}^{\infty}\left(\int_0^t F_i(\tau)\xi_j(\tau)e_i d\tau\right)H_{\varepsilon_{n(i,j)}} \in S(H)_{-0}\ .$$

If $F(t)$ is a deterministic $\mathbb{R}$-valued function, then we have

$$\int_0^t F(\tau)\delta W(\tau) = \sum_{i=1}^{\infty}\int_0^t F(\tau)e_i\delta\beta_i(\tau)\,.$$

Example 6 Consider a Hitsuda-Skorohod integrable process $F(t) : \mathbb{R} \to S(H)_{-0}$ with form

$$F = \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^{\infty} c_{i\alpha}H_\alpha e_i = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha = \sum_{i=1}^{\infty} F_i e_i \in S(H)_{-0}\,, \quad c_\alpha \in H,\ F_i \in (S)_{-0}\,.$$

We define the Hitsuda-Skorohod integral of F_i with respect to the Brownian motion β_i by

$$\int_0^t F_i(\tau)\delta\beta_i(\tau) := \sum_{\gamma_i\in\mathcal{J}}\sum_{\alpha+\varepsilon_{n(i,j)}=\gamma_i}\left(\int_0^t c_{i\alpha}(\tau)\xi_j(\tau)d\tau\right)H_{\gamma_i}\,.$$

Then the Hitsuda-Skorohod integral of F can be written in the form

$$\begin{aligned}
\int_0^t F(\tau)\diamond\mathbb{W}(\tau)d\tau &= \sum_{\gamma\in\mathcal{J}}\left(\sum_{i=1}^{\infty}\sum_{\alpha+\varepsilon_k=\gamma}\int_0^t c_{i\alpha}(\tau)\kappa_{ik}(\tau)e_i\right)H_\gamma \\
&= \sum_{i=1}^{\infty}\sum_{\gamma_i\in\mathcal{J}}\sum_{\alpha+\varepsilon_{n(i,j)}=\gamma_i}\left(\int_0^t c_{i\alpha}(\tau)\xi_j(\tau)e_i d\tau\right)H_{\gamma_i} \\
&= \sum_{i=1}^{\infty}\int_0^t F_i(\tau)e_i\delta\beta_i(\tau)\,.
\end{aligned}$$

The Hermite Transform

$H_{\mathbb{C}}$ will denote the complexification of H.

Definition 7 *For* $q \in \mathbb{R}_+$*, define the infinite-dimensional neighborhoods of* 0 *in* $\mathbb{C}^{\mathbb{N}}$ *as*

1. $\mathbb{K}_q^n := \left\{ z \in (\mathbb{C}^{\mathbb{N}})_c \ ; \ |z_i| < (2i)^{-q}, \ i \le n \text{ and } z_i = 0, \ i > n \right\}$
   $\mathbb{K}_q := \left\{ z \in \mathbb{C}^{\mathbb{N}} \ ; \ |z_i| < (2i)^{-q}, \ i \in \mathbb{N} \right\}$.
2. $\overline{\mathbb{K}}_q^n := \left\{ z \in (\mathbb{C}^{\mathbb{N}})_c \ ; \ |z_i| \le (2i)^{-q}, \ i \le n \text{ and } z_i = 0, \ i > n \right\}$
   $\overline{\mathbb{K}}_q := \left\{ z \in \mathbb{C}^{\mathbb{N}} \ ; \ |z_i| \le (2i)^{-q}, \ i \in \mathbb{N} \right\}$.

We can regard $\mathbb{K}_q^n$ and $\overline{\mathbb{K}}_q^n$ as subsets of $\mathbb{C}^n$.

Definition 8 *Define the Hermite transform of* $F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega) \in S(H)_{-1}$ *as*

$$\mathcal{H}F(z) = \widetilde{F}(z) := \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha \ ,$$

for $z \in \mathbb{C}^{\mathbb{N}}$ *so that limit exists in* $H_{\mathbb{C}}$*.*

Proposition 10 *For* $F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \in S(H)_{-1}$*, there exists a* $q \in \mathbb{N} \setminus \{1\}$ *so that for all* $z \in \overline{\mathbb{K}}_q$*,* $\mathcal{H}(F)(z)$ *converges absolutely and*

$$\sum_{\alpha \in \mathcal{J}} \|c_\alpha\|_{H_{\mathbb{C}}} |z^\alpha| \le \|F\|_{-1,-q} \ (A(q))^{1/2} \ .$$

Here

$$A(q) := \sum_{\alpha \in \mathcal{J}} (2\mathbb{N})^{-q\alpha}$$

converges if and only if $q > 1$ [20].

Proposition 11 *Consider* $X(z) : \mathbb{K}_q \to H_{\mathbb{C}}$*,* $q \in \mathbb{N} \setminus \{1\}$ *having form*

$$X(z) = \sum_{\alpha \in \mathcal{J}} c_\alpha z^\alpha, \quad c_\alpha \in H \ ,$$

bounded by some $M < \infty$*. Then the formal sum*

$$F(\omega) := \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega) \ ,$$

belongs to $S(H)_{-1,-4q}$ *and hence to* $S(H)_{-1}$*. Also* $\mathcal{H}F(z) = X(z)$ *and* $\|F(\omega)\|_{-1,-4q} \le MA(q)$*.*

Proposition 12 **(a)** *Consider* $F, \ G \in S(H)_{-1}$*. Then for all* z *such that both* $\mathcal{H}F_i(z)$ *and* $\mathcal{H}G_i(z)$ *exist*

$$\mathcal{H}(F \diamond G)(z) = \sum_{i=1}^{\infty} \mathcal{H}F_i(z) \ \mathcal{H}G_i(z) \ e_i \ .$$

(b) *Consider* $F \in (S)_{-1}, G \in S(H)_{-1}$*. Then for all* z *such that both* $\mathcal{H}F(z)$ *and* $\mathcal{H}G(z)$ *exist*

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \ \mathcal{H}G(z) \ .$$

Definition 9 *Consider* $F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \in S(H)_{-1}$. *Define the generalized expectation of* F *as*

$$E[F] := c_{(0,\ldots)} = \widetilde{F}(0) \in H \ .$$

If $F \in L^p,\ p > 1$, then this coincides with the usual expectation. Clearly we have for all $F, G \in S(H)_{-1}$

$$E[F \diamond G] = \sum_{i=1}^{\infty} E[F_i]\ E[G_i]\ e_i \ ,$$

and for all $F \in (S)_{-1}$ and $G \in S(H)_{-1}$

$$E[F \diamond G] = E[F]\ E[G] \ .$$

3 Main Results

For a bounded operator B on H, define the action of B on any element $F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha$ belonging to $S(H)_{\pm\rho},\ \rho \in [0,1]$ or $L^2(H)$ as

$$(BF)(\omega) := \sum_{\alpha \in \mathcal{J}} (Bc_\alpha) H_\alpha(\omega) \ .$$

Since $\|Bc_\alpha\|_H \leq \|B\|\ \|c_\alpha\|_H$, BF is an element of the same space as F.

Now let A be an unbounded operator on H with $\mathcal{D}(A) \subseteq H$. We now define action of A on $S(H)_{-1}$.

Definition 10 *Define the domain of* A *in* $S(H)_{-1}$ *as*

$$\mathcal{D}(A)_{-1} := \left\{ F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \in S(H)_{-1} \ : \sum_{\alpha \in \mathcal{J}} \|Ac_\alpha\|_H^2 (2\mathbb{N})^{-q\alpha} < \infty \right\} \ , \quad \textit{for some } q \in \mathbb{N} \ ,$$

and the action of A *on* $F \in \mathcal{D}(A)_{-1}$ *by*

$$(AF)(\omega) := \sum_{\alpha \in \mathcal{J}} (Ac_\alpha) H_\alpha(\omega) \ .$$

Similarly we can define action of A on any $S(H)_{\pm\rho},\ \rho \in [0,1]$ with the corresponding domains $\mathcal{D}(A)_{\pm\rho}$. If there is no confusion about the space where operator A acts, we will write simply $\mathcal{D}(A)$ for the domain of A.

Lemma 4 *If* A *is a closed operator on* H *and* $F \in \mathcal{D}(A)_{-1}$, *then there exists a* $q \in \mathbb{N} \setminus \{1\}$ *such that* $\widetilde{F} \equiv \mathcal{H}F(z)$ *and* $\mathcal{H}(AF)(z)$ *exist for all* $z \in \mathbb{K}_q$ *and*

$$\mathcal{H}(AF)(z) = A\widetilde{F}(z) \ , \ z \in \mathbb{K}_q \ .$$

Generalized Stochastic Convolution

Let $\{Y(t),\ t \geq 0\}$ be a family of bounded strongly continuous linear operators on H. For any $F(t) \in S(H)_{-1}$ we define the Wick product

$$Y(t) \diamond F(t) := Y(t)F(t) = Y(t)\sum_{i=1}^{\infty} F_i(t)e_i = \sum_{i=1}^{\infty} F_i(t)Y(t)e_i = \sum_{\alpha\in\mathcal{J}} (Y(t)c_\alpha)H_\alpha \ . \qquad (4)$$

Now if $Y(t-s)\mathbb{W}(s)$ is Pettis integrable on $[0,t]$, satisfying lemma 3, the integral

$$\begin{aligned}\int_0^t Y(t-s)\delta W(s) \ &:= \ \int_0^t Y(t-s) \diamond \mathbb{W}(s)ds = \int_0^t Y(t-s)\mathbb{W}(s)ds \\ &= \ \sum_{k=1}^{\infty}\left(\int_0^t Y(t-s)\kappa_k(s)ds\right) H_{\varepsilon_k} = \sum_{i=1}^{\infty}\int_0^t Y(t-s)e_i\delta\beta_i(s) \ ,\end{aligned}$$

is called the *generalized stochastic convolution.* We obviously have that its generalized expectation is equal to zero.

Proposition 13 *Suppose that for some* $T \in (0,\infty)$

$$\int_0^T \|Y(t)\|^2 dt < \infty \ .$$

Then $Y(t)\mathbb{W}(t)$ *is Pettis integrable on* $[0,T]$ *and for all* $t \in [0,T]$

$$\int_0^t Y(s)\delta W(s) = \sum_{k=1}^{\infty}\left(\int_0^t Y(s)\kappa_k(s)ds\right) H_{\varepsilon_k}(\omega) \in S(H)_{-0} \ .$$

Note that since

$$\sum_{k=1}^{\infty}\left\|\int_0^t Y(t-s)\kappa_k(s)ds\right\|_H^2 \leq C\sum_{k=1}^{\infty}\int_0^t \|Y(t-s)e_k\|_H^2\, ds \ ,$$

then under the additional assumption

$$\sum_{k=1}^{\infty}\int_0^t \|Y(t-s)e_k\|_H^2\, ds < \infty \ ,$$

or equivalently, that the linear operator

$$L_t x := \int_0^t Y(s)Y^*(s)x ds \ , \ x \in H \ ,$$

is of trace class, we obtain that the generalized stochastic convolution is an element of $L^2(H)$. This agrees with the results of [5] , where it is also shown that in this case the generalized stochastic convolution is a Gaussian random variable with mean 0 and covariance operator L_t. In this case we have the equality

$$\int_0^t Y(s)\delta W(s) = \sum_{i=1}^{\infty}\int_0^t Y(s)e_i d\beta_i(s),$$

where

$$\int_0^t Y(s)e_i d\beta_i(s) = \lim_{N\to\infty}\sum_{k=1}^{N} Y(t_k^*)e_i\left[\beta_i(t_{k+1}) - \beta_i(t_k)\right]$$

is the usual Ito integral and the limit is in $L^2(H)$.

Stochastic Differential Equations

Consider the following differential equation in $S(H)_{-1}$

$$\frac{dX(t)}{dt} = AX(t) + B\mathbb{W}(t), \quad t \in [0,T], \quad X(0) = X_0 \in \mathcal{D}(A)_{-1} \subseteq S(H)_{-1} \ , \tag{5}$$

where A is the generator of a C_0-semigroup $\{S(t),\ t \geq 0\}$ on H and B is a bounded operator on H.

We say that continuously differentiable process $X(\cdot)$ is a $S(H)_{-1}$ solution of (5) if $X(t) \in \mathcal{D}(A)_{-1}$ for all $t \in [0,T]$ and satisfies the equation for $t \in [0,T]$.

Taking Hermite transforms in (5), we obtain the equation

$$\frac{d\widetilde{X}(t,z)}{dt} = A\widetilde{X}(t,z) + B\widetilde{\mathbb{W}}(t,z) \quad t \in [0,T], \quad \widetilde{X}(0) = \widetilde{X_0} \in \mathcal{D}(A) \subseteq H \ . \tag{6}$$

The function

$$\widetilde{X}(t,z) = S(t)\widetilde{X_0}(z) + \int_0^t S(t-s)B\widetilde{\mathbb{W}}(s,z)ds \ .$$

is a solution of (6) for $z \in \mathbb{K}_q$, for $q \geq 4$. This implies that the process

$$X(t) = S(t)X_0 + \int_0^t S(t-s)B\mathbb{W}(s)ds = S(t)X_0 + \int_0^t S(t-s)B\delta W(s) \tag{7}$$

satisfies (5) in $S(H)_{-1}$. Here the generalized stochastic convolution $\int_0^t S(t-s)B\delta W(s) =: W_A(t)$ is an element of $S(H)_{-0}$, and we have $E[W_A(t)] = 0$ and $E[X(t)] = E[S(t)X_0] = S(t)E[X_0]$.

Thus, we have the following result.

Theorem 1 *Suppose that A is the generator of a C_0-semigroup $\{S(t),\ t \geq 0\}$ on H and B is a bounded linear operator on H. If $X_0 \in S(H)_{-1}$, then the stochastic differential equation (5) has a unique continuously differentiable $S(H)_{-1}$ solution*

$$X(t) = S(t)X_0(\omega) + \int_0^t S(t-s)B\delta W(s) \ , \quad X_0 \in \mathcal{D}(A) \ ,$$

with $E[X(t)] = S(t)E[X_0]$.

Corollary 3 *If $X_0 \notin \mathcal{D}(A)$, then the process (7) is the unique continuous $S(H)_{-1}$ solution for the equation*

$$X(t) = X_0 + A\int_0^t X(s)ds + BW(t) \ .$$

Theorem 2 *If a process $F(t) : \mathbb{R} \to S(H)_{-0}$ is Pettis integrable, then the process*

$$X(t) = S(t)X_0(\omega) + \int_0^t S(t-s)F(s)ds + \int_0^t S(t-s)B\delta W(s) \ , \quad X_0 \in \mathcal{D}(A) \ ,$$

is a unique continuously differentiable $S(H)_{-1}$ solution of the equation

$$\frac{dX(t)}{dt} = \Big[AX(t) + F(t)\Big] + B\mathbb{W}(t), \quad t \in [0,T], \quad X(0) = X_0.$$

Proof follows from (4), lemma 3 and theorem 1.

Second Order Term Structure Models

To illustrate the white noise approach to second order term structure models we consider the simplest case of constant volatility. Then the dynamics of the deformation process is described by the following equation

$$\begin{aligned} dY(t,\xi) &= \left[\frac{\partial Y}{\partial \xi}(t,\xi) + \frac{\kappa}{2}\frac{\partial^2 Y}{\partial \xi^2}(t,\xi)\right]dt + \sigma_0 dW(t), \quad t \geq 0,\ \xi \in [0,M], \\ Y(0,\xi) &= Y_0(\xi), \end{aligned} \tag{8}$$

on the separable Hilbert space $H = L^2_\kappa(0,M)$ with the norm

$$\|f\|^2_\kappa = \int_0^M e^{\frac{2\xi}{\kappa}} |f(\xi)|^2 d\xi\,.$$

The operator

$$A = \frac{\partial}{\partial \xi} + + \frac{\kappa}{2}\frac{\partial^2}{\partial \xi^2}$$

with the domain $\mathcal{D}(A) = H^2(0,M) \cap H^1_0(0,M)$, where $H^2(0,M)$ and $H^1_0(0,M)$ are the following Sobolev spaces

$$\begin{aligned} H^1_0(0,M) &= \left\{u \in L^2_\kappa(0,M) \;\Big|\; \frac{\partial u}{\partial \xi} \in L^2_\kappa(0,M),\ u(t,0) = u(t,M) = 0\right\}, \\ H^2(0,M) &= \left\{u \in L^2_\kappa(0,M) \;\Big|\; \frac{\partial^2 u}{\partial \xi^2} \in L^2_\kappa(0,M)\right\}, \end{aligned}$$

is self-adjoint on H. The eigenfunctions and eigenvalues of $-A$ can be obtained by solving

$$\frac{\kappa}{2}\frac{d^2 e_n}{d\xi^2} + \frac{de_n}{d\xi} = -\mu_n e_n, \quad e_n(0) = e_n(M) = 0,\ n \in \mathbb{N},$$

which gives

$$\mu_n = \frac{1}{2\kappa}\Big(1 + \frac{n^2\pi^2\kappa^2}{M^2}\Big) > 0, \quad e_n = \sqrt{\frac{2}{M}}\, e^{-\frac{\xi}{\kappa}} \sin\frac{n\pi\xi}{M},\ n \in \mathbb{N}.$$

Since $\{e_n\}_{n=1}^\infty$ forms an orthonormal basis in H, any $f \in H$ can be written in the form

$$f = \sum_{n=1}^\infty f_n e_n, \quad \text{where} \quad f_n = \langle f, e_n\rangle_H,$$

and we have

$$\|f\|_H = \Big(\sum_{n=1}^\infty |f_n|^2\Big)^{1/2}.$$

For each $t \geq 0$, define a bounded linear operator on H by

$$S(t)v := \sum_{n=1}^\infty e^{-\mu_n t} v_n e_n, \quad v \in H.$$

The operators S form a C_0-semigroup $\{S(t),\ t \geq 0\}$ whose generator is A. Therefore the process

$$\begin{aligned} Y(t) &= S(t)Y_0 + \int_0^t S(t-s)\sigma_0 \delta W(s) \\ &= \sum_{n=1}^\infty \Big(e^{-\mu_n t} Y_{0n} e_n + \sigma_0 \int_0^t e^{-\mu_n(t-s)} e_n d\beta_n(s)\Big). \end{aligned}$$

is the unique solution to equation (8). We also note here that this solution belongs to $L^2(H)$.

HJM Model in Musiela Parametrization

Consider the Hilbert space $H = L^2_\gamma(0,\infty)$ of measurable functions with

$$\|x\|^2_\gamma = \int_0^\infty e^{-\gamma\xi}|x|^2 d\xi < \infty.$$

The general HJM equation in Musiela parametrization with infinite dimensional noise can be written in the form:

$$\begin{aligned} &\frac{dX(t)}{dt} = AX(t) + F(t) + \sigma(t) \diamond \mathbb{W}(t), \quad t \geq 0, \\ &X(0) = X_0, \end{aligned} \tag{9}$$

where F and σ are $S(H)_{-0}$ processes, $\mathbb{W}$ is H-valued white noise. The operator $A = \frac{\partial}{\partial\xi}$ with the domain

$$\mathcal{D}(A) = \Big\{ u \in L^2_\gamma(0,\infty) \;\Big|\; \frac{\partial u}{\partial \xi} \in L^2_\gamma(0,\infty) \Big\},$$

is the generator of a C_0-semigroup $\{S(t),\ t \geq 0\}$ defined by

$$S(t)f(\xi) = f(t+\xi), \quad t,\xi \geq 0$$

on H. If F is Pettis integrable and $\sigma = \sum_{n=1}^\infty \sigma_n e_n$ is Hitsuda-Skorohod integrable, then the unique $S(H)_{-0}$ solution to equation (9) is given by

$$\begin{aligned} X(t) &= S(t)X_0 + \int_0^t S(t-s)F(s)ds + \int_0^t S(t-s)\Big(\sigma(s) \diamond \mathbb{W}(s)\Big)ds \\ &= S(t)X_0 + \int_0^t S(t-s)F(s)ds + \sum_{n=1}^\infty \int_0^t S(t-s)\sigma_n(s)e_n \delta\beta_n(s)\,. \end{aligned}$$

Note that under the assumption of absence of an arbitrage strategy one has that $F \equiv F_{HJM} = \sum_{n=1}^\infty \mathcal{S}\sigma_n$, see [10], [8] for details. We also note that under some additional assumptions on σ and F (for example see assumptions in [10], [8]) the process X constructed above will belong to $L^2(H)$.

4 Final Remarks

1. In the final version of this paper we will also discuss the HJM, the LIBOR and the second order term structure models in the case of multiplicative infinite dimensional noise, i.e. functions σ, F, Γ, b will depend on the unknown X, e.g. $\sigma = \sigma(t,\xi,X)$, etc.

2. In our worthcoming paper *White noise approach to stochastic invariance and consistency of financial models* we use the white noise technique to address the problem of consistency of these models.

3. The second order term structure models can be modified to involve higher order derivatives of Y, so the operator A will not necessarily be the generator of a C_0-semigroup. Our approach allows one to solve the corresponding equations in the case when A is the generator of an integrated semigroup (see [9]).

4. It is noted in [5] that the stochastic integration with respect to a Brownian sheet can be reformulated in terms of stochastic integration with respect to a cylindrical Wiener process ($d \geq 2$).

References

[1] T. Björk and B.J. Christensen, *Interest rate dynamics and consistent forward rate curves*, Mathematical Finance, **9** (1999), no. 4, 323–348.

[2] A. Brace, D. Gątarek and M. Musiela, *The market model of interest rate dynamics*, Mathematical Finance, **7** (1997), no.2, 127–155.

[3] J.P. Bouchaud, R. Cont, N.El. Karoui, M. Potters and N. Sagna, *Phenomenology of the interest rate curve: a statistical analysis of the term structure deformations*, Working paper, 1997.

[4] R. Cont, *Modelling term structure dynamics: an infinite dimensional approach*, International Journal of Theoretical and Applied Finance, to appear.

[5] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[6] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite-Dimensional Systems*, Cambridge University Press, Cambridge, 1996.

[7] R. Douady, *Yield curve smoothing and residual variance of fixed income positions*, Working paper, Courant Institute of Math. Sciences, New York University, 1997.

[8] B. Goldys and M. Musiela, *Infinite dimensional diffusions, Kolmogorov equations and interest rate models.* In *Advances in Mathematical Finance: Topics in Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, Cambridge, 2001.

[9] A. Filinkov and J. Sorensen, *Differential equations in spaces of abstract stochastic distributions*, Stochastics and Stochastics Reports, to appear.

[10] D. Filipović *Consistency Problems for Heath-Jarrow-Morton Interest Rate Models*, Springer-Verlag, Berlin-Heidelberg, 2001.

[11] D. Heath, R. Jarrow and A. Morton, *Bond pricing and the term structure of interest rates: a new methodology*, Econometrica, **60** (1992), 77–105.

[12] T. Hida, *Brownian motion*, Springer-Verlag, New York, 1980.

[13] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit, *White Noise*, Kluwer Academic Publishers, Dordrecht, 1993.

[14] H. Holden, B. Øksendal, J. Ubøe and T. Zhang, *Stochastic Partial Differential Equations*, Birkhäuser, Boston, 1996.

[15] H.-H. Kuo, *White Noise Distribution Theory*, CRC Press, Boca Raton, 1996.

[16] M. Musiela, *Stochastic PDEs and term structure models*, Journées Internationales de Finance, IGR-AFFI, La Baule, 1993.

[17] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, Applications of Mathematics, vol. 36, Springer-Verlag, Berlin-Heidelberg, 1997.

[18] P. Santa-Clara and D. Sornette, *The dynamics of the forward interest rate curve with stochastic string shocks*, Working paper, University of California, Los Angeles, 1999.

[19] J. Zabczyk, *Stochastic invariance and consistency of financial models*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., **11** (2000), no. 2, 67–80.

[20] T. Zhang, *Characterizations of white noise test functions and Hida distributions*, Stochastics Stochastics Rep., **41**(1992), 71–87.