# Testing for Conditional Heteroscedasticity in Financial Time-Series

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**Abstract**

In this paper, we survey time-series models allowing for conditional heteroscedasticity and autoregression, like AR-GARCH-type models. These models reduce to a white noise model, when some of the conditional heteroscedasticity parameters take their boundary value at zero, and the autoregressive component is in fact not present. We reproduce the asymptotic distribution of the pseudo-log-likelihood ratio statistics for testing the present conditional heteroscedasticity models on reduction to white noise. The theoretical results are applied to financial data, i.e. log-returns of stock prices. We estimate the parameters for all models presented and further on, we test on reduction to white noise. The impact of these results on risk measurement is discussed by comparing Value-at-Risk calculations under alternative model specifications.

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Key words: Conditional Heteroscedasticity, GARCH-Models, Autoregression, Pseudo-Likelihood Ratio Test, Value-at-Risk, Risk Measurement.

## 1 Introduction

The classical Black-Scholes model turned out to be a very effective basis for risk management and for the valuation of derivative securities, see Hull (1993) and Jorion (2001). In recent years, a class of discrete time processes which allow for conditional heteroscedasticity received growing attention. The ARCH and GARCH models were introduced by Engle (1982) and Bollerslev (1986). Their impact on both Value-at-Risk calculation and option pricing is well understood and has been topic of empirical investigations, see Bollerslev et al. (1992), Duan (1995), and Frey and McNeil (2000).

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Conditional heteroscedasticity models incorporate the Black-Scholes model[1] as a special case and therefore, they provide a more general framework than white noise models. But the generality of the conditional heteroscedasticity models demands more sophisticated methods in Value-at-Risk calculation and option pricing and of course more computational effort. For this reason, testing conditional heteroscedasticity models for reduction to the Black-Scholes model becomes an important subject. From the point of view of mathematical statistics, testing on model reduction is a challenging task, since standard tests like Lagrange multiplier test and likelihood ratio tests in their general form fail. The reason for this is that the null hypothesis of conditional homoscedasticity corresponds to a boundary value of the parameter space with respect to the general model. Nevertheless, this topic has been studied in recent years and results on various conditional heteroscedasticity models have been carried out, see Demos and Sentana (1998), Andrews (1999b), and Klüppelberg et al. (2000).

In this paper, we survey time-series models allowing for conditional heteroscedasticity and autoregression. In particular, we study the ARCH(1), GARCH(1,1), and AR(1)-GARCH(1,1) model. These models reduce to white noise, i.e. the Black-Scholes model, when some of the conditional heteroscedasticity parameters take their boundary value at zero, and the autoregressive component is in fact not present. We state the asymptotic distribution of pseudo-log-likelihood ratio statistics for testing the presented conditional heteroscedasticity models for reduction to white noise.

The theoretical results studied in this paper are applied to financial data, i.e. log-returns of stock prices. We estimate the parameters for all models presented and further on, we test on reduction to white noise. The empirical observations indicate whether the time-series exhibits conditional heteroscedasticity or the data corresponds to white noise and therefore, the Black-Scholes framework is appropriate. We show examples where the test accepts the model reduction and hence, the more feasible Black-Scholes framework is sufficient. The impact of these results on risk measurement is discussed by comparing Value-at-Risk calculations under alternative model specifications, i.e. the conditional heteroscedasticity model and the Black-Scholes approach.

The paper is organized as follows: Section 2 gives a survey on conditional heteroscedasticity models allowing for autoregression including the asymptotic distribution of the likelihood ratio statistics for testing on conditional homoscedasticity. Estimation and testing results are given in Section 3. In Section 4, the impact of the estimation and testing results on Value-at-Risk calculation is discussed. Section 5 concludes and gives an outlook.

[1] We remark, that in this paper, we name those models "Black-Scholes model" that are of *iid* type, i.e. the log-returns of the asset price process are *iid*, implying that the accumulated return process is a random walk. Hence, the *iid* models are given by the class of discrete time Lévy processes and not only by a discrete time Brownian motion like the term "Black-Scholes model" would suggest.

## 2 AR-GARCH Models and the LR Statistics

In this section, time-series models allowing for conditional heteroscedasticity and autoregression are presented. Additionally, for each model, we reproduce the form of the Maximum Likelihood Estimator (MLE), and the asymptotic distribution of the Likelihood Ratio (LR) statistics for testing on conditional homoscedasticity. This is carried out for an AR-GARCH model studied by Klüppelberg et al. (2000), and the well known ARCH and GARCH models (see Bera and Higgins, 1993, and Bollerslev, 1986), where in the two latter cases testing on conditional homoscedasticity is discussed in Demos and Sentana (1998) and Andrews (1999b).

First, we specify the probabilistic setting where the time-series is placed in and recall the form of the maximum likelihood estimator and the deviance for testing on model reduction in a general framework. Let $(\Omega, P, \mathcal{F}, (\mathcal{F}_t)_{t\geq 1})$ be a filtered probability space in discrete time. The innovations driving the time-series are given by an i.i.d. family of random variables $(\varepsilon_t)_{t\geq 2}$ with zero expectation and unit variance, and a finite fourth moment. The filtration $(\mathcal{F}_t)$ is the natural filtration of the innovation process $(\varepsilon_t)$.

The time-series $(X_t)_{t\geq 1}$ with initial value $X_1 \in \mathbf{R}$ is defined by

$$X_t = \alpha X_{t-1} + \sigma_t \varepsilon_t, \quad \text{for } t = 2, 3..., \tag{1}$$

where $\alpha \in [-1, 1]$ and $(\sigma_t)_{t\geq 1}$ is a positive predictable process, i.e. $\sigma_t$ is $\mathcal{F}_{t-1}$–measurable. With $e_t = \sigma_t \varepsilon_t$ for $t \geq 2$, we can write Equation (1) in the form

$$X_t = \alpha X_{t-1} + e_t, \quad \text{for } t = 2, 3\dots. \tag{2}$$

Thus, the process $(X_t)_{t\geq 1}$ is autoregressive with innovations $(e_t)_{t\geq 2}$ showing conditional variance $\mathbf{E}\left\{e_t^2 \,|\mathcal{F}_{t-1}\right\} = \sigma_t^2$, for $t \geq 2$.

The pseudo-log likelihood function for a finite sample of length $T \in \mathbf{N}$ is given by

$$\mathcal{L}_T(\theta) = -\frac{1}{2}\sum_{t=2}^{T} \ln \sigma_t^2 - \frac{1}{2}\sum_{t=2}^{T} \varepsilon_t^2 - \frac{1}{2}(T-1)\ln(2\pi), \tag{3}$$

where $\theta$ is a vector describing the model, hence $\theta$ parametrizes $\sigma_t$ and $\varepsilon_t$ in Equation (3).

For testing purposes, we assume that the *true model* is given by $\theta \in \Omega$. For a given subset $\tau$ of $\Omega$, we can test the null hypothesis $\theta \in \tau$ versus the alternative $\theta \in \Omega \setminus \tau$. The test utilized in this paper is the likelihood ratio test, therefore, we define the deviance

$$d_T = -2\left(\mathcal{L}_T(\hat{\theta}_0) - \mathcal{L}_T(\hat{\theta})\right), \tag{4}$$

where $\hat{\theta}_0 \in \tau$ is the maximum likelihood estimator on the null hypothesis, and $\hat{\theta}$ is the corresponding estimator on the alternative $\Omega \setminus \tau$. Later on, we specify the conditional heteroscedasticity models and reproduce the asymptotic distribution of the deviance statistics. It is worth mentioning that the derivation of the asymptotic distribution of the deviance statistics is a non-trivial mathematical task. Testing on conditional homoscedasticity transfers to the problem of testing a boundary hypothesis, since the conditional heteroscedasticity parameters take their boundary value at zero in the conditional homoscedastic case.

Klüppelberg et al. (2001) discuss conditional heteroscedasticity models allowing also for autoregression as given in Equation (1) by specifying a AR(1)-GARCH(1,1) model. The conditional variance of the innovations is determined by

$$\sigma_t^2 = \beta + \lambda e_{t-1}^2 + \delta\sigma_{t-1}^2 = \beta + \lambda\sigma_{t-1}^2\varepsilon_{t-1}^2 + \delta\sigma_{t-1}^2, \qquad \text{for } t = 2, 3..., \tag{5}$$

where $\beta > 0$ and $\lambda \geq 0$ and $\delta \geq 0$, and $\theta = (\alpha, \beta, \lambda, \delta) \in \Omega = [-1, 1] \times \mathbf{R}^+ \times \mathbf{R}_0^+ \times \mathbf{R}_0^+$. The log likelihood function reads as in Equation (3) with

$$\sigma_t^2 = \vartheta + \delta^{t-1}(\sigma_1^2 - \vartheta) + \lambda \sum_{i=1}^{t-1} \delta^{i-1} \left(X_{t-i} - \alpha X_{t-1-i}\right)^2, \quad \text{and} \tag{6}$$

$$\varepsilon_t^2 = (X_t - \alpha X_{t-1})^2 \left(\vartheta + \delta^{t-1}(\sigma_1^2 - \vartheta) + \lambda \sum_{i=1}^{t-1} \delta^{i-1} \left(X_{t-i} - \alpha X_{t-1-i}\right)^2\right)^{-1} \tag{7}$$

for $t = 3, ...,$ where $\vartheta = \frac{\beta}{1-\delta}$, and $\sigma_1 > 0$, and $\sigma_2^2 = \beta + \delta\sigma_1^2$ and $\varepsilon_2^2 = \frac{(X_2 - \alpha X_1)^2}{\beta + \delta\sigma_1^2}$.

The null hypothesis of conditional homoscedasticity and the absence of autoregression is given by the set $\tau = \{0\} \times \mathbf{R}^+ \times \{0\} \times \{0\}$, i.e. $\alpha = \lambda = \delta = 0$. For $\theta \in \tau$, the log likelihood function stated in Equations (6) and (7) simplifies to

$$\sigma_t^2 = \beta \quad \text{and} \quad \varepsilon_t^2 = \frac{X_t^2}{\beta}, \qquad \text{for } t = 2, 3..., \tag{8}$$

Klüppelberg et al. (2001) computed the asymptotic distribution of the deviance statistics for testing the null hypothesis $\tau$ versus the alternative $\Omega \setminus \tau$. Under the null $\tau$,

$$d_T \xrightarrow{D} N^2 + Z^2 \mathbf{1}_{\{Z \geq 0\}}, \quad \text{for } T \to \infty, \tag{9}$$

where $\mu_3$ and $\mu_4$ are the third and fourth moment of the innovations $(\varepsilon_t)_{t \geq 2}$, and

$$Z = \frac{\mu_3^2}{\sqrt{2(\mu_4 - 1)}} N + \sqrt{\frac{(\mu_4 - 1)^2 - \mu_3^4}{2(\mu_4 - 1)}}\, \widetilde{N}$$

with $N$ and $\widetilde{N}$ independent standard normal random variables.

Assuming the innovations $(\varepsilon_t)_{t \geq 2}$ are standard normal, yields $\mu_3 = 0$ and $\mu_4 = 3$. The asymptotic distribution of the deviance statistics becomes

$$d_T \xrightarrow{D} N^2 + \widetilde{N}^2 \mathbf{1}_{\{\widetilde{N} \geq 0\}}, \quad \text{for } T \to \infty, \tag{10}$$

where $N$ and $\widetilde{N}$ are independent standard normal random variables.

We remark that the hypothesis of conditional homoscedasticity is formulated by $\lambda = 0$ and $\delta = 0$. However, if $\lambda = 0$, the variance process $(\sigma_t^2)_{t\geq 1}$ is a deterministic function converging to $\vartheta = \frac{\beta}{1-\delta}$. For sufficiently large sample size, conditional homoscedasticity appears though $\delta > 0$. Accordingly, conditional homoscedasticity can be specified by $\lambda = 0$ and $\delta \geq 0$. This causes a nuisance parameter $\delta$ to appear that cannot be identified under the null hypothsesis. Andrews (1999b) covers this problem for GARCH(1,1). Here, we restrict ourselves to the approach considered by Klüppelberg et al. (2000), hence the hypothesis of conditional homoscedasticity is given by $\lambda = 0$ and $\delta = 0$.

An approach frequently applied for modeling financial time-series is a GARCH(1,1) model, see Bollerslev (1986). Within this framework, the conditional heteroscedasticity is specified, but no autoregression is taken into account, hence $\alpha = 0$, and Equation (1) reduces to $X_t = \sigma_t \varepsilon_t$ for $t \geq 2$. The conditional heteroscedasticity is given by

$$\sigma_t^2 = \beta + \lambda e_{t-1}^2 + \delta \sigma_{t-1}^2, \qquad \text{for } t = 2, 3..., \tag{11}$$

where $\sigma_1^2 > 0$ is given, and $\beta > 0$, $\lambda \geq 0$, and $\delta \geq 0$, and $\theta = (\beta, \lambda, \delta) \in \Omega = \mathbf{R}^+ \times \mathbf{R}_0^+ \times \mathbf{R}_0^+$. By setting $\vartheta = \frac{\beta}{1-\delta}$, the log likelihood function in Equation (3) is determined by

$$\sigma_t^2 = \vartheta + \delta^{t-1}(\sigma_1^2 - \vartheta) + \lambda \sum_{i=1}^{t-1} \delta^{i-1} X_{t-i}^2, \quad \text{and} \tag{12}$$

$$\varepsilon_t^2 = X_t^2 \left( \vartheta + \delta^{t-1}(\sigma_1^2 - \vartheta) + \lambda \sum_{i=1}^{t-1} \delta^{i-1} X_{t-i}^2 \right)^{-1}, \quad \text{for } t = 2, ..., \tag{13}$$

Following the approach we presented for the AR(1)-GARCH(1,1) model, the null hypothesis of conditional homoscedasticity could read $\lambda = 0$ and $\delta = 0$. As mentioned before, the parameter $\delta$ appears to be a nuisance parameter for this formulation of the null hypothesis, since we can not identify $\lambda$ and $\delta$ simultaneously under the null. Andrews (1999b) shows a way to control this problem. In his framework, he applies stationarity arguments and therefore, he assumes $\delta \in \Delta$ apriori, where $\Delta = [0, \delta_u]$ with $\delta_u < 1$. With this assumption, the parameter space $\Omega$ is of the form $\Omega = \mathbf{R}^+ \times \mathbf{R}_0^+ \times \Delta$. Furthermore, he formulates the null hypothesis of conditional homoscedasticity by $\lambda = 0$, hence $\tau = \mathbf{R}^+ \times \{0\} \times \Delta$.

On the parameter space $\Omega$ describing the alternative, the information matrix becomes singular under the null hypothesis, hence we cannot identify $\lambda$ and $\delta$ simultaneously. Andrews (1999b) overcomes this problem by fixing $\delta \in \Delta$ in a first step, i.e. the parameter space is restricted to $\Omega_\delta = \mathbf{R}^+ \times \mathbf{R}_0^+ \times \{\delta\}$, for each $\delta \in \Delta$. On each restricted space $\Omega_\delta$, a maximum likelihood estimation is carried out—what is now possible, since $\delta$ is fixed. This results into $\mathcal{L}_T(\hat{\theta}_\delta; \delta)$, where $\hat{\theta}_\delta$ is the maximizer of the log likelihood function on $\Omega_\delta$. In a second step, the supremum is taken over all $\delta \in \Delta$, and Equation (4) becomes

$$\mathcal{L}_T(\hat{\theta}) = \sup_{\delta \in \Delta} \mathcal{L}_T(\hat{\theta}_\delta; \delta), \tag{14}$$

where $\hat{\theta}$ is the maximizing argument—that needs not to be unique. When the initial condition is $\sigma_1 = \vartheta = \frac{\beta}{1-\delta}$, the log likelihood does not depend on $\delta$ for any $\theta \in \tau$. Hence, the estimator on the hypothesis $\tau$ is still given by Equation (8) and does not depend on the nuisance parameter $\delta$—at least asymptotically, for large $T$ and arbitrary $\delta \in \Delta$. With this specification, Andrews (1999b) obtains the asymptotic distribution of the deviance statistics $d_T$ under the null hypothesis.

$$d_T \stackrel{D}{\to} \frac{\mu_4 - 1}{2} \sup_{\delta \in \Delta} Z_\delta^2 \mathbf{1}_{\{Z_\delta \geq 0\}}, \quad \text{for } T \to \infty, \tag{15}$$

where $\mu_4$ is the fourth moment of the innovations and $(Z_\delta)_{\delta \in \Delta}$ is a Gaussian process with covariance structure

$$\operatorname{cov}(\mathrm{Z}_{\delta_1}, \mathrm{Z}_{\delta_2}) = \frac{(1-\delta_1^2)(1-\delta_2^2)}{1-\delta_1\delta_2}, \quad \text{for } \delta_1, \delta_2 \in \Delta.$$

For computational purposes, we can write Equation (15) as

$$d_T \stackrel{D}{\to} c \sup_{\delta \in \Delta} Y_\delta^2 \mathbf{1}_{\{Y_\delta \geq 0\}}, \quad \text{for } T \to \infty, \tag{16}$$

where $c = \frac{\mu_4 - 1}{2}$ and $Y_\delta = \sqrt{1-\delta^2} \sum_{i=0}^{\infty} \delta^i \widetilde{Z}_i$, with $\left(\widetilde{Z}_i\right)_{i \geq 0}$ are *iid* standard normal random variables. Furthermore, we can replace $c$ by the estimator $\hat{c}_T$, where

$$\hat{c}_T = \frac{1}{2} \left( \frac{\frac{1}{T}\sum_{t=1}^T X_t^4}{\left(\frac{1}{T}\sum_{t=1}^T X_t^2\right)^2} - 1 \right), \tag{17}$$

and define a rescaled test statistics

$$\frac{d_T}{\hat{c}_T} \stackrel{D}{\to} \sup_{\delta \in \Delta} Y_\delta^2 \mathbf{1}_{\{Y_\delta \geq 0\}}, \quad \text{for } T \to \infty, \tag{18}$$

where the asymptotic distribution under the null hypothesis is preserved. Andrews (1999b) generated the asymptotic critical values by simulation. For $\Delta = [0, 0.95]$, for significance levels 10%, 5%, and 1%, the critical values are 3.06, 4.33, and 7.30 respectively.

Finally, we consider the ARCH(1) model. Properties of this model, and estimation and testing are surveyed in Bera and Higgins (1993). The conditional variance is specified by

$$\sigma_t^2 = \beta + \lambda e_{t-1}^2, \quad \text{for } t = 2, 3..., \tag{19}$$

where $\beta > 0$ and $\lambda \geq 0$, and $\theta = (\beta, \lambda) \in \Omega = \mathbf{R}^+ \times \mathbf{R}_0^+$. The log likelihood function in Equation (3) is determined by

$$\sigma_t^2 = \beta + \lambda X_{t-1}^2, \quad \text{and} \tag{20}$$

$$\varepsilon_t^2 = \frac{X_t^2}{\beta + \lambda X_{t-1}^2}, \quad \text{for } t = 2, \dots. \tag{21}$$

The null hypothesis of conditional homoscedasticity is given by $\lambda = 0$, or $\tau = \mathbf{R}^+ \times \{0\}$. The maximum likelihood estimator on the null hypothesis is given by Equation (8). The asymptotic distribution of the deviance for testing $\tau$ versus $\Omega$ is deduced in Demos and Sentana (1998) for Gaussian innovations.

$$d_T \xrightarrow{D} N^2 \mathbf{1}_{\{N\geq 0\}}, \quad \text{for } T \to \infty, \tag{22}$$

where $N$ is standard normal. This result can be generalized for non Gaussian innovations by setting $\Delta = \{0\}$ in the GARCH(1,1) model—see Equation (15), and hence Equation (22) becomes

$$d_T \xrightarrow{D} \frac{\mu_4 - 1}{2} N^2 \mathbf{1}_{\{N\geq 0\}}, \quad \text{for } T \to \infty. \tag{23}$$

Applying the result of Andrews (1999b) for the case $\Delta = 0$, we can rescale the deviance by $c_T = \frac{\mu_4 - 1}{2}$. The estimate of $c_T$ is given by $\hat{c}_T$ in Equation (17).

$$\frac{d_T}{\hat{c}_T} \xrightarrow{D} N^2 \mathbf{1}_{\{N\geq 0\}}, \quad \text{for } T \to \infty, \tag{24}$$

with critical values of 1.64, 2.71, and 5.41 corresponding to significance levels of 10%, 5%, and 1% respectively, see Demos and Sentana (1998).

| Alternative | 10 % | 5% | 1 % |
|---|---|---|---|
| ARCH(1) ($d^1$) | 1.64 | 2.71 | 5.41 |
| GARCH(1) ($d^2$) | 3.06 | 4.33 | 7.30 |
| AR(1)-GARCH(1,1) ($d^3$) | 3.80 | 5.13 | 8.28 |

Table 1: Critical values for testing the Black-Scholes model vs. various alternatives.

The required critical values for testing purposes are given in Table 1. We remark that the statistics $d^1$ and $d^2$ are rescaled statistics, hence the asymptotic distribution does not depend on the characteristics of the innovations. Unfortunately, the LR statistics $d^3$ for testing white noise versus the AR(1)-GARCH(1,1) alternative cannot be rescaled. The critical values depend on the third and fourth moment of the innovation process, see Equation (9). In Table 1, the critical values for $d^3$ are listed for "normal" innovations, i.e. $\mu_3 = 0$ and $\mu_4 = 3$. In the "non-normal" situation, the critical values have to be computed by simulation in each individual case. In this paper, we use the standardized residuals for estimating $\mu_3$ and $\mu_4$ of the innovation process.

Using the theoretical results presented in this section, we analyze log-returns of stock prices observed at the European market and the US market. The statistical analysis includes parameter estimation and testing for conditional homoscedasticity. Furthermore, the impact on applications in finance is discussed, where we focus on Value-at-Risk calculation.

## 3 Parameter Estimation and Testing

Choosing an appropriate model is an important and difficult task—not only for applications like Value-at-Risk calculation. In this section, we compare the models presented in

Section 2 empirically. In particular, we examine log-returns of stock prices observed at the German market and the US market for conditional heteroscedasticity. We estimate the parameters of the models, and proceed by testing for reduction to the Black-Scholes model. We explicitly show the impact of the size of the alternative on the test result. The more alternatives are offered, the more likely is the rejection of the null hypothesis.

In the following, we analyze the daily log-returns of Allianz, BASF, Deutsche Telekom, VW, Apple, and IBM. The observed time period ranges from Sep. 1, 1996 to Sep. 1, 2000—with exception of Deutsche Telekom that was first listed Nov. 18, 1996. This includes 1044 data points for the entire 4 year horizon. In addition, we examine the most recent 2 years and the final year of the given time horzion, including 523 data points and 262 data points respectively. Detailed estimation results are reported in the Appendix. We focus on testing the null hypothesis of white noise, where the critical values are reported in Table 1.

| Test results (Allianz) | $\widehat{d^1}$ | $\widehat{d^2}$ | $\widehat{d^3}$ |
|---|---|---|---|
| 1996 - 2000 | 15.24 | 24.37 | 97.61 |
| 1998 - 2000 | 6.76 | 9.20 | 42.53 |
| 1999 - 2000 | 7.52 | 7.64 | 50.25 |
| Test results (VW) | $\widehat{d^1}$ | $\widehat{d^2}$ | $\widehat{d^3}$ |
| 1996 - 2000 | 16.43 | 43.16 | 87.57 |
| 1998 - 2000 | 13.45 | 32.71 | 64.88 |
| 1999 - 2000 | 8.77 | 8.77 | 15.90 |

Table 2: Deviance and rescaled deviance statistics for Allianz and VW. $^{*}/^{**}/^{***}$ denotes acceptance of the null hypothesis for the significance level of 1/5/10%.

The log-returns of Allianz and VW show strong evidence for conditional heteroscedasticity. We observe significant ARCH and GARCH effects, see Table 7 and Table 8 in the Appendix. The low standard errors indicate that the data fits into the time-series framework. The interpretation of the estimation results is validated by the test on model reduction to white noise versus various alternatives. For all investigated time horizons and both stocks—Allianz and VW, the null hypothesis of white noise is rejected for all admissible alternatives, see Table 2.

The test results of Deutsche Telekom and Apple are given in Table 3. Deutsche Telekom shows conditional heteroscedasticity. For the 1-year and 2-year horizon, we estimate a low ARCH effect that is not even very significant. However, the GARCH coefficient appears to be important, since the likelihood clearly improves for the model enhanced by the GARCH parameter $\delta$, see Table 9 in the Appendix. Accordingly, we expect the result of the test for reduction to white noise to depend significantly on the set of given alternatives. The ARCH(1) alternative is not matching the conditional heteroscedasticity effects of Deutsche Telekom, hence the null hypothesis of white noise is accepted even for the 10% significance level. The GARCH(1,1) model provides the more appropriate set of alternatives. Here, the null hypothesis of white noise is clearly rejected in all cases. The same holds of course for the AR(1)-GARCH(1,1) alternative.

| Test results (D. Telekom) | $\widehat{d^1}$ | $\widehat{d^2}$ | $\widehat{d^3}$ |
|---|---|---|---|
| 1996 - 2000 | 8.71 | 86.15 | 177.35 |
| 1998 - 2000 | 0.67*** | 19.17 | 31.81 |
| 1999 - 2000 | 0.11*** | 7.82 | 12.56 |
| Test results (Apple) | $\widehat{d^1}$ | $\widehat{d^2}$ | $\widehat{d^3}$ |
| 1996 - 2000 | 4.87* | 12.41 | 49.33 |
| 1998 - 2000 | 4.48* | 5.54* | 13.65 |
| 1999 - 2000 | 1.33*** | 1.72*** | 6.85* |

Table 3: Deviance and rescaled deviance statistics for Deutsche Telekom and Apple. */**/*** denotes acceptance of the null hypothesis for the significance level of 1/5/10%.

For Apple, we also notice that the acceptance or rejection of the white noise null hypothesis is influenced by the set of alternatives. The ARCH parameter $\lambda$ is slightly significant, whereas $\delta$ is estimated with a remarkable high standard error, especially for the 1-year and 2-year horizon, see Table 9. Thus, the hypothesis of white noise tends to be rejected when the set of alternatives captures autoregression. This fact becomes apparent particularly for the 1-year horizon, where the likelihood increases substantially when introducing the autoregression parameter.

| Test results (BASF) | $\widehat{d^1}$ | $\widehat{d^2}$ | $\widehat{d^3}$ |
|---|---|---|---|
| 1996 - 2000 | 10.08 | 17.08 | 27.32 |
| 1998 - 2000 | 0.32*** | 5.28* | 7.32* |
| 1999 - 2000 | 0.00*** | 0.13*** | 1.29*** |
| Test results (IBM) | $\widehat{d^1}$ | $\widehat{d^2}$ | $\widehat{d^3}$ |
| 1996 - 2000 | 2.00** | 4.46* | 15.58* |
| 1998 - 2000 | 0.00*** | 0.04*** | 1.05*** |
| 1999 - 2000 | 0.00*** | 0.03*** | 0.17*** |

Table 4: Deviance and rescaled deviance statistics for BASF and IBM. */**/*** denotes acceptance of the null hypothesis for the significance level of 1/5/10%.

Finally, we observe the BASF and IBM data, see Table 4. The null hypothesis of white noise cannot be rejected for almost all time horizons and significance levels. Thus BASF and IBM are standard examples for log-returns of Black-Scholes type. For this kind of data, parameter estimation becomes complicated, since the information matrix is asymptotically singular for the presented models incorporating GARCH effects, i.e. $\delta \geq 0$—see the discussion in Andrews (1999b). If the data is white noise, we have to apply the procedure proposed in Section 2. For fixed $\delta$, we maximize the likelihood function, and this is carried out for $\delta \in \Delta$, where we of course choose a finite set, i.e. $\delta \in \{0, 0.01, ..., 0.95\}$. We take the supremum of the maximized likelihood function depending on $\delta$ and compute the deviance statistics. In this case, the parameter $\delta$ is reported with no standard error of course, since it is more a nuisance parameter than an estimate, see Table 11 and Table 12.

Nevertheless, we are able to run the maximum likelihood estimation procedure for some data close to *iid*, despite of the theoretical and also numerical problems that result from an (almost) singular information matrix—e.g. Apple, see Table 10.

Dealing with "white noise" data, the numerical procedure often overextends the estimation tools for GARCH of standard software packages. Brooks et al. (2001) discuss the accuracy of GARCH(1,1) model estimation—even in a "friendly setting". In this paper, we computed the Maximum Likelihood by a Newton-Raphson scheme, where we use the analytic gradient and Hessian matrix, what is close to the benchmark given by Brooks et al. (2001) in the sense of estimation accuracy.

The discussed LR tests are appropriate methods for model choice, but the computation of the asymptotic distribution of the deviance may become challenging, what was shown in Section 2. Besides, there exist other (weaker) criteria for selecting a model in the "best" way. Akaike's Information Criterion (AIC) is the most commonly used and is given by

$$\mathrm{AIC} = -2\,\mathcal{L}(\hat{\theta}) + 2\,p\,, \tag{25}$$

where $\mathcal{L}(\hat{\theta})$ is the maximized log likelihood funtion and $p$ denotes the number of parameters, see Chatfield (2001). We cross-check the LR test results with respect to the AIC, see Table 6. AIC prefers the Black-Scholes model exactly, when the LR-test results accepts the null hypothesis of white noise on the 10% level. In all other cases AIC suggests to choose the alternative time-series model subject to the LR test.

In the following section, we employ the results carried out here. Especially, we study the impact of model choice—of course within the presented framework—on Value-at-Risk calculation, where we are not only concerned with the VaR qualitity in terms of prediction accuracy, but also tackle the issue of computability of the estimates.

# 4 VaR under different Model Specifications

The focus of this application is to illustrate the test results of the previous section by studying the performance of the different models with respect to Value-at-Risk calculation. Here, VaR calculation is a one-day prediction of a conditional quantile for a fixed level $\gamma$. Generally, the quality of the VaR calculation can be measured by standard backtesting according to Basle, see Jorion (2001). By this, we are enabled to evaluate each model due to the backtesting result. On the other hand, the Likelihood Ratio test results in Section 3 indicate which model to choose for fitting the data most adequately. In this section, the task is to compare the results of the backtesting procedure and the likelihood ratio test.

For each log-return series analyzed in Section 3, we perform a standard backtest. We use a 500-day history to estimate the parameter of each specific model in order to calculate the one-day VaR prediction on the 99% level. For the time-series, we now assume normal distributed innovations, hence the $\gamma$-VaR is given by

$$\mathrm{VaR}(\gamma) = -\mu_{t+1} + \sigma_{t+1}\Phi^{-1}(\gamma)\,, \tag{26}$$

where $\Phi$ is the standard normal distribution function, and $\mu_{t+1}$ is the mean value prediction, and $\sigma_{t+1}$ the standard deviation prediction both based on the preceding 500 observations $X_t, ..., X_{t-499}$, and $\gamma = 99\%$. This is carried out for the last 500 days within the sample period, and for that period, we count the number of VaR exceptions. The Basle traffic light evaluates the backtesting result—the number of exceptions—by assigning "Green", "Yellow" or "Red". For the 99% level, the Green Zone ranges from 0 to 8, the Yellow Zone from 9 to 14, and the Red Zone starts with 15.

| | AR-GARCH | GARCH | ARCH | Black-Scholes |
|---|---|---|---|---|
| Allianz | 4 (G) / 5.637% | 4 (G) / 5.663% | 5 (G) 5.677% | 5 (G) / 5.706% |
| VW | 5 (G) / 5.395% | 6 (G) / 5.434% | 4 (G) **5.888%** | 3 (G) / **6.037%** |
| D. Telekom | 5 (G) / 7.309% | 5 (G) / 7.316% | **11 (Y)** 6.471% | **12 (Y)** / 6.393% |
| Apple | **10 (Y)** / 8.739% | 8 (G) / 8.760% | 8 (G) 8.795% | 8 (G) / 8.774% |
| BASF | 5 (G) / 4.685% | 4 (G) / 4.690% | 4 (G) 4.681% | 4 (G) / 4.685% |
| IBM | **7 (G)** / 5.387% | **8 (G)** / 5.392% | 6 (G) 5.255% | 6 (G) / 5.255% |

Table 5: Backtesting results, i.e. number of exceptions including traffic light according to Basle (Green,Yellow,Red) and average Value-at-Risk.

The backtesting result is given in Table 5. With respect to VaR calculation, the number of exceptions together with the Basle traffic light characterize the quality of the model from the regulator's point of view. As well, we report the average VaR. A competing interest of financial institutions is to minimize the VaR as much as possible, since they have to keep a certain amount of their own capital proportional to the VaR. Roughly speaking, we examine each model for its risk in the sense of Basle and for its cost, where we interpret cost as own capital requirement.

For data with non negligible conditional heteroscedasticity effects—Allianz, VW, Deutsche Telekom—the backtesting results suggest to choose the more complex models like AR(1)-GARCH(1,1) and GARCH(1,1). In the case of VW, the number of exceptions is equal to 5 for all models, but the price in form of the average VaR increases considerably for the more simple models, e.g. the average VaR of the Black-Scholes model exceeds the average VaR of AR(1)-GARCH(1,1) by 12%. The Black-Scholes model and the ARCH(1) model have a significantly lower average VaR for Deutsche Telekom, however they also exhibits a clear "Yellow" traffic light with 11 and 12 exceptions.

Reviewing the test results for Apple, BASF, and IBM—the data close to white noise—the more simple Black-Scholes and the ARCH(1) model should be chosen. The average VaR attains for all models approximately the same value for each stock, but the number of exceptions tends to increase for the more complex models. The larger number of exceptions for the models incorporating the GARCH-component arises primarily from the numerical problems within the estimation procedure. For data close to white noise, the information matrix may become singular and consequently, the MLE is not reliable. The estimation procedure occasionally creates artificial and misleading effects that result in poor VaR predictions. For data close to white noise, the more complex models involving a GARCH-component are not advisable.

# 5 Summary and Outlook

This paper compares financial time-series models allowing for conditional heteroscedasticity and autoregression. Primarily, we utilize the likelihood ratio test for the comparison of the different models, and cross-check the LR result by applying the AIC concept, and also, we perform standard backtests according to Basle. In general, we can not find evidence for preferring a specific model for all observed log-returns. We are suggested to use the more simple Black-Scholes model for data close to white noise, and we ought to rely on GARCH-type models, whenever the data exhibits conditional heteroscedasticity. This result is striking, especially in the case of backtesting, since the largest model—the AR(1)-GARCH(1,1) model—incorporates all other models discussed in this paper. And hence, we would not expect heterogeneous test results, since the largest model should cover all possible effects. The reason for this can be found in the numerics of the estimation, i.e. the information matrix becomes singular when we apply GARCH models to white noise data. Finally, we believe that it is not possible to find a "benchmark model" for describing financial time-series. In the end, the problem of model choice has to be discussed in each specific case.

The presented LR tests are sophisticated statistical tools. In contrast to the AIC, the LR test allows for a calculation of the power function—at least, this can be carried out by a simulation study. This issue should be topic of further research, particularly for the AR(1)-GARCH(1,1) model when testing for reduction to white noise. Furthermore, the results of Klüppelberg et al. (2000) offer the opportunity of testing for mean-reversion. This is also an important subject, especially for interest rates.

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| AIC for Allianz | Black-Scholes | ARCH | GARCH | AR-GARCH |
|---|---|---|---|---|
| 1996 - 2000 | -4922.32 | -4980.38 | -5014.34 | -5013.94 |
| 1998 - 2000 | -2363.00 | -2391.78 | -2400.86 | -2399.54 |
| 1999 - 2000 | -1197.80 | -1241.98 | -1240.70 | -1242.04 |
| AIC for VW | Black-Scholes | ARCH | GARCH | AR-GARCH |
| 1996 - 2000 | -4898.76 | -4937.00 | -4978.46 | -4980.32 |
| 1998 - 2000 | -2422.44 | -2444.92 | -2477.96 | -2481.32 |
| 1999 - 2000 | -1318.24 | -1331.86 | -1329.86 | -1328.14 |
| AIC for D. Telekom | Black-Scholes | ARCH | GARCH | AR-GARCH |
| 1996 - 2000 | -4315.64 | -4331.22 | -4485.32 | -4487.00 |
| 1998 - 2000 | -2092.38 | -2091.40 | -2117.48 | -2118.18 |
| 1999 - 2000 | -1018.66 | -1016.82 | -1025.44 | -1025.24 |
| AIC for Apple | Black-Scholes | ARCH | GARCH | AR-GARCH |
| 1996 - 2000 | -3873.46 | -3889.60 | -3915.64 | - 3916.80 |
| 1998 - 2000 | -1909.60 | -1913.28 | -1912.64 | -1917.24 |
| 1999 - 2000 | -913.86 | -913.38 | -911.82 | -914.72 |
| AIC for BASF | Black-Scholes | ARCH | GARCH | AR-GARCH |
| 1996 - 2000 | -5267.96 | -5282.04 | -5287.22 | -5289.28 |
| 1998 - 2000 | -2597.50 | -2595.94 | -2600.80 | -2598.80 |
| 1999 - 2000 | -1308.96 | -1306.96 | -1305.10 | -1304.10 |
| AIC for IBM | Black-Scholes | ARCH | GARCH | AR-GARCH |
| 1996 - 2000 | -4920.84 | -4925.62 | -4931.94 | -4930.42 |
| 1998 - 2000 | -2350.28 | -2348.28 | -2346.40 | -2345.32 |
| 1999 - 2000 | -1130.74 | -1128.74 | -1126.84 | -1124.90 |

Table 6: AIC for Allianz, VW, Deutsche Telekom, Apple, BASF, and IBM.

| Sep. 1., 1996 - Sep. 1., 2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
|---|---|---|---|---|---|
| Black-Scholes | - | 0.5213 | - | - | 2462.16 |
| s.e. | (-) | (0.0230) | (-) | (-) | |
| ARCH(1) | - | 0.4125 | 0.2101 | - | 2492.19 |
| s.e. | (-) | (0.0220) | (0.0449) | (-) | |
| GARCH(1,1) | - | 0.0740 | 0.1630 | 0.7062 | 2510.17 |
| s.e. | (-) | (0.0098) | (0.0340) | (0.0073) | |
| AR(1)-GARCH(1,1) | 0.0456 | 0.0714 | 0.1629 | 0.7113 | 2510.97 |
| s.e. | (0.0365) | (0.0110) | (0.0374) | (0.0196) | |
| Sep. 1., 1998 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 0.6308 | - | - | 1182.50 |
| s.e. | (-) | (0.0394) | (-) | (-) | |
| ARCH(1) | - | 0.4863 | 0.1379 | - | 1197.89 |
| s.e. | (-) | (0.0390) | (0.0690) | (-) | |
| GARCH(1,1) | - | 0.1961 | 0.2359 | 0.4779 | 1203.43 |
| s.e. | (-) | (0.0624) | (0.0782) | (0.1303) | |
| AR(1)-GARCH(1,1) | 0.0493 | 0.1747 | 0.2240 | 0.5213 | 1203.77 |
| s.e. | (0.0636) | (0.0620) | (0.0666) | (0.1193) | |
| Sep. 1., 1999 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 0.5904 | - | - | 599.90 |
| s.e. | (-) | (0.0511) | (-) | (-) | |
| ARCH(1) | - | 0.3112 | 0.6509 | - | 622.99 |
| s.e. | (-) | (0.0414) | (0.1755) | (-) | |
| GARCH(1,1) | - | 0.2865 | 0.6509 | 0.0430 | 623.35 |
| s.e. | (-) | (0.0589) | (0.1559) | (0.0608) | |
| AR(1)-GARCH(1,1) | -0.1253 | 0.2823 | 0.7377 | 0.0130 | 625.02 |
| s.e. | (0.0630) | (0.0428) | (0.1790) | (0.0350) | |

Table 7: Parameter estimates with standard errors (s.e.) in parentheses, and the maximum log likelihood (ML) for Allianz.

| Sep. 1., 1996 - Sep. 1., 2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
|---|---|---|---|---|---|
| Black-Scholes | - | 0.5356 | - | - | 2450.38 |
| s.e. | (-) | (0.0234) | (-) | (-) | |
| ARCH(1) | - | 0.4412 | 0.1761 | - | 2470.50 |
| s.e. | (-) | (0.0138) | (0.0320) | (-) | |
| GARCH(1,1) | - | 0.0178 | 0.0665 | 0.9024 | 2492.23 |
| s.e. | (-) | (0.0053) | (0.0120) | (0.0178) | |
| AR(1)-GARCH(1,1) | 0.0654 | 0.0180 | 0.0664 | 0.9020 | 2494.16 |
| s.e. | (0.0314) | (0.0054) | (0.0125) | (0.0184) | |
| Sep. 1., 1998 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 0.5679 | - | - | 1212.22 |
| s.e. | (-) | (0.0351) | (-) | (-) | |
| ARCH(1) | - | 0.4391 | 0.2348 | - | 1224.46 |
| s.e. | (-) | (0.0283) | (0.0562) | (-) | |
| GARCH(1,1) | - | 0.0136 | 0.0577 | 0.9157 | 1241.98 |
| s.e. | (-) | (0.0067) | (0.0142) | (0.0230) | |
| AR(1)-GARCH(1,1) | 0.1022 | 0.0115 | 0.0522 | 0.9246 | 1244.66 |
| s.e. | (0.0421) | (0.0060) | (0.0136) | (0.0211) | |
| Sep. 1., 1999 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 0.3721 | - | - | 660.12 |
| s.e. | (-) | (0.0326) | (-) | (-) | |
| ARCH(1) | - | 0.2881 | 0.2132 | - | 667.93 |
| s.e. | (-) | (0.0321) | (0.0835) | (-) | |
| GARCH(1,1) | - | 0.2881 | 0.2132 | 0.0000 | 667.93 |
| s.e. | (-) | (0.0351) | (0.0841) | (0.0481) | |
| AR(1)-GARCH(1,1) | 0.1083 | 0.2892 | 0.2045 | 0.0000 | 668.07 |
| s.e. | (0.0708) | (0.0499) | (0.0849) | (0.1238) | |

Table 8: Parameter estimates with standard errors (s.e.) in parentheses, and the maximum log likelihood (ML) for VW.

| Sep. 1., 1996 - Sep. 1., 2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
|---|---|---|---|---|---|
| Black-Scholes | - | 0.7439 | - | - | 2158.82 |
| s.e. | (-) | (0.0335) | (-) | (-) | |
| ARCH(1) | - | 0.6417 | 0.1446 | - | 2167.61 |
| s.e. | (-) | (0.0230) | (0.0380) | (-) | |
| GARCH(1,1) | - | 0.0067 | 0.0678 | 0.9264 | 2245.66 |
| s.e. | (-) | (0.0036) | (0.0111) | (0.0104) | |
| AR(1)-GARCH(1,1) | 0.0652 | 0.0066 | 0.0702 | 0.9244 | 2247.50 |
| s.e. | (0.0361) | (0.0021) | (0.0115) | (0.0106) | |
| Sep. 1., 1998 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 1.0675 | - | - | 1047.19 |
| s.e. | (-) | (0.0660) | (-) | (-) | |
| ARCH(1) | - | 1.0175 | 0.0480 | - | 1047.70 |
| s.e. | (-) | (0.0585) | (0.0466) | (-) | |
| GARCH(1,1) | - | 0.0116 | 0.0362 | 0.9530 | 1061.74 |
| s.e. | (-) | (0.0072) | (0.0114) | (0.0135) | |
| AR(1)-GARCH(1,1) | 0.0739 | 0.0115 | 0.0370 | 0.9522 | 1063.09 |
| s.e. | (0.0488) | (0.0071) | (0.0115) | (0.0135) | |
| Sep. 1., 1999 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 1.1903 | - | - | 510.33 |
| s.e. | (-) | (0.1040) | (-) | (-) | |
| ARCH(1) | - | 1.1583 | 0.0281 | - | 510.41 |
| s.e. | (-) | (0.1039) | (0.0609) | (-) | |
| GARCH(1,1) | - | 0.2300 | 0.1328 | 0.6766 | 515.72 |
| s.e. | (-) | (0.1670) | (0.0700) | (0.2002) | |
| AR(1)-GARCH(1,1) | 0.0901 | 0.2049 | 0.1220 | 0.7064 | 516.62 |
| s.e. | (0.0753) | (0.1607) | (0.0659) | (0.1931) | |

Table 9: Parameter estimates with standard errors (s.e.) in parentheses, and the maximum log likelihood (ML) for Deutsche Telekom.

| Sep. 1., 1996 - Sep. 1., 2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
|---|---|---|---|---|---|
| Black-Scholes | - | 1.4301 | - | - | 1937.73 |
| s.e. | (-) | (0.0625) | (-) | (-) | |
| ARCH(1) | - | 1.2671 | 0.1215 | - | 1946.80 |
| s.e. | (-) | (0.0299) | (0.0368) | (-) | |
| GARCH(1,1) | - | 0.4133 | 0.1175 | 0.5943 | 1960.82 |
| s.e. | (-) | (0.0994) | (0.0343) | (0.0918) | |
| AR(1)-GARCH(1,1) | -0.0635 | 0.4150 | 0.1188 | 0.5911 | 1962.40 |
| s.e. | (0.0365) | (0.1003) | (0.0344) | (0.0932) | |
| Sep. 1., 1998 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 1.5114 | - | - | 955.80 |
| s.e. | (-) | (0.0936) | (-) | (-) | |
| ARCH(1) | - | 1.3133 | 0.1379 | - | 958.64 |
| s.e. | (-) | (0.0957) | (0.0690) | (-) | |
| GARCH(1,1) | - | 0.8283 | 0.1330 | 0.3247 | 959.32 |
| s.e. | (-) | (0.3426) | (0.0657) | (0.2517) | |
| AR(1)-GARCH(1,1) | -0.1262 | 0.9665 | 0.1375 | 0.2214 | 962.62 |
| s.e. | (0.0495) | (0.3183) | (0.0650) | (0.2294) | |
| Sep. 1., 1999 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 1.7757 | - | - | 457.93 |
| s.e. | (-) | (0.1551) | (-) | (-) | |
| ARCH(1) | - | 1.5909 | 0.1076 | - | 458.69 |
| s.e. | (-) | (0.1719) | (0.0978) | (-) | |
| GARCH(1,1) | - | 1.1339 | 0.1225 | 0.2450 | 458.91 |
| s.e. | (-) | (0.6179) | (0.0987) | (0.3718) | |
| AR(1)-GARCH(1,1) | -0.1508 | 1.4602 | 0.0832 | 0.0782 | 461.36 |
| s.e. | (0.0672) | (0.8076) | (0.0915) | (0.4603) | |

Table 10: Parameter estimates with standard errors (s.e.) in parentheses, and the maximum log likelihood (ML) for Apple.

| Sep. 1., 1996 - Sep. 1., 2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
|---|---|---|---|---|---|
| Black-Scholes | - | 0.3761 | - | - | 2634.98 |
| s.e. | (-) | (0.0165) | (-) | (-) | |
| ARCH(1) | - | 0.3345 | 0.1087 | - | 2643.02 |
| s.e. | (-) | (0.0153) | (0.0312) | (-) | |
| GARCH(1,1) | - | 0.0346 | 0.0602 | 0.8484 | 2646.61 |
| s.e. | (-) | (0.0133) | (0.0162) | (0.0475) | |
| AR(1)-GARCH(1,1) | -0.0087 | 0.0342 | 0.0601 | 0.8491 | 2648.64 |
| s.e. | (0.0325) | (0.0136) | (0.0164) | (0.0485) | |
| Sep. 1., 1998 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 0.4025 | - | - | 1299.75 |
| s.e. | (-) | (0.0249) | (-) | (-) | |
| ARCH(1) | - | 0.3886 | 0.0353 | - | 1299.97 |
| s.e. | (-) | (0.0317) | (0.0559) | (-) | |
| GARCH(1,1) | - | 0.0111 | 0.0001 | 0.9500 | 1303.40 |
| s.e. | (-) | (0.0033) | (0.0082) | (fixed) | |
| AR(1)-GARCH(1,1) | 0.0327 | 0.0114 | 0.0000 | 0.9500 | 1303.40 |
| s.e. | (0.0445) | (0.0085) | (0.0119) | (fixed) | |
| Sep. 1., 1999 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 0.3857 | - | - | 655.48 |
| s.e. | (-) | (0.0338) | (-) | (-) | |
| ARCH(1) | - | 0.3857 | 0.0000 | - | 655.48 |
| s.e. | (-) | (0.0478) | (0.0881) | (-) | |
| GARCH(1,1) | - | 0.1870 | 0.0277 | 0.4900 | 655.55 |
| s.e. | (-) | (0.0320) | (0.0751) | (fixed) | |
| AR(1)-GARCH(1,1) | 0.0755 | 0.0198 | 0.0004 | 0.9500 | 656.10 |
| s.e. | (0.0622) | (0.0285) | (0.0191) | (fixed) | |

Table 11: Parameter estimates with standard errors (s.e.) in parentheses, and the maximum log likelihood (ML) for BASF.

| Sep. 1., 1996 - Sep. 1., 2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
|---|---|---|---|---|---|
| Black-Scholes | - | 0.5244 | - | - | 2461.42 |
| s.e. | (-) | (0.0230) | (-) | (-) | |
| ARCH(1) | - | 0.4941 | 0.0589 | - | 2464.81 |
| s.e. | (-) | (0.1245) | (0.0209) | (-) | |
| GARCH(1,1) | - | 0.0249 | 0.0298 | 0.9237 | 2468.97 |
| s.e. | (-) | (0.0083) | (0.0084) | (0.0204) | |
| AR(1)-GARCH(1,1) | -0.0251 | 0.0341 | 0.0294 | 0.9237 | 2469.21 |
| s.e. | (0.0325) | (0.0085) | (0.0087) | (0.0210) | |
| Sep. 1., 1998 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 0.6464 | - | - | 1176.14 |
| s.e. | (-) | (0.0400) | (-) | (-) | |
| ARCH(1) | - | 0.6486 | 0.0000 | - | 1176.14 |
| s.e. | (-) | (0.0420) | (0.0198) | (-) | |
| GARCH(1,1) | - | 0.3746 | 0.0000 | 0.4300 | 1176.20 |
| s.e. | (-) | (0.0261) | (0.0196) | (fixed) | |
| AR(1)-GARCH(1,1) | -0.0557 | 0.4696 | 0.0000 | 0.2800 | 1176.66 |
| s.e. | (0.0444) | (0.0375) | (0.0195) | (fixed) | |
| Sep. 1., 1999 - Sep. 1.,2000 | $\widehat{\alpha}$ | $\widehat{\beta}$ $[10^{-3}]$ | $\widehat{\lambda}$ | $\widehat{\delta}$ | ML |
| Black-Scholes | - | 0.7633 | - | - | 566.37 |
| s.e. | (-) | (0.0668) | (-) | (-) | |
| ARCH(1) | - | 0.7655 | 0.0000 | - | 566.37 |
| s.e. | (-) | (0.0704) | (0.0292) | (-) | |
| GARCH(1,1) | - | 0.5022 | 0.0000 | 0.3600 | 566.42 |
| s.e. | (-) | (0.0488) | (0.0317) | (fixed) | |
| AR(1)-GARCH(1,1) | -0.1173 | 0.4787 | 0.0000 | 0.3900 | 566.45 |
| s.e. | (0.0632) | (0.0525) | (0.0314) | (fixed) | |

Table 12: Parameter estimates with standard errors (s.e.) in parentheses, and the maximum log likelihood (ML) for IBM.