Perfect Hedging of Index Derivatives under a Locally Arbitrage Free Minimal Market Model

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October 12, 2001

Abstract. The paper presents a financial market model that generates stochastic volatility using a minimal set of factors. These factors, formed from transformations of square root processes, model the dynamics of different denominations of a benchmark portfolio. Benchmarked prices are assumed to be local martingales. Numerical results for the pricing and hedging of basic derivatives on indices are described. This includes cases where the standard risk neutral pricing methodology fails. However, payoffs can be perfectly hedged using self-financing strategies and a form of arbitrage still exists. This is illustrated by hedge simulations. The term structure of implied volatilities is documented.

1991 *Mathematics Subject Classification:* primary 90A12; secondary 60G30, 62P20.
JEL Classification: G12
Key words and phrases: derivative pricing, arbitrage, minimal market model.

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1 Introduction

The standard approach currently used to price and hedge derivatives is based on a pricing methodology that specifies an equivalent martingale measure. This immediately restricts financial modelers to a class of asset price dynamics, which excludes almost all forms of arbitrage. However, this assumption may be too strong to capture the idiosyncracies of existing markets that, as they mature, may converge towards arbitrage free markets. In addition, markets may be imperfect following shocks or turbulence. In Platen (2001a, 2001b) a benchmarked pricing methodology was proposed, which assumes benchmarked prices to be local martingales. This approach allows some form of arbitrage, but enforces the perfect hedging of derivatives using self-financing portfolios.

This paper studies examples of derivative pricing and hedging for a specific diffusion model, the *minimal market model* (MMM) proposed in Platen (2000a, 2000b). It uses a minimal number of factors and for certain parameter choices permits a form of arbitrage. The factors are modeled as transformations of square root processes under the real world probability measure. Basic building blocks of the MMM, are the different denominations of the *growth optimal portfolio* (GOP) measured in units of the different primary assets. They determine key financial quantities including the short rates, volatilities and risk premia. The MMM generates endogenously stochastic volatility without the requirements of introducing any additional stochastic volatility process.

This paper demonstrates how to price and hedge basic index derivatives under the MMM in the presence of some form of arbitrage without relying on the standard risk neutral pricing methodology. In particular, we describe examples, where arbitrage is present under the model. Furthermore, the implied volatility term structures that arise for European call and put options on indices are obtained. Finally, we present simulations, where standard derivatives are perfectly hedged but where a form of arbitrage still exists.

The paper is organized as follows. Section 2 describes the MMM. In Section 3 the price of European style contingent claims is derived. Some basic index derivatives are studied in Section 4. Section 5 shows how to construct self-financing hedging strategies. Finally, hedge simulations are presented in Section 6.

2 Minimal Market Model

2.1 Growth Optimal Portfolio

We consider the evolution of the prices of $d+1$ *primary tradeable assets*, $d \in \{1, 2, \dots\}$, in a market which can be, for instance, currencies or stocks. These are modeled on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$. Here the filtration

$\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ fulfills the usual conditions with $\mathcal{A}_0$ being trivial, see Karatzas & Shreve (1988).

The time value of the domestic currency is expressed via the corresponding *savings account process* $B^0 = \{B^0(t),\, t \in [0,T]\}$, where

$$dB^0(t) = f^0(t)\, B^0(t)\, dt$$

for $t \in [0,T]$, $T \in (0,\infty)$, with $B^0(0) = 1$. This savings account accumulates interest continuously according to the *domestic short rate process* $f^0 = \{f^0(t),\, t \in [0,T]\}$ and describes the shortest instantaneous forward rate for holding the domestic currency. In our notation, the 0th primary asset is interpreted as the domestic currency. The time value of the jth primary asset is similarly modeled by the jth *savings account process* $B^j = \{B^j(t),\, t \in [0,T]\}$, where

$$dB^j(t) = B^j(t)\, f^j(t)\, dt \tag{2.1}$$

for $t \in [0,T]$ with $B^j(0) = 1$, $j \in \{1,2,\dots,d\}$. The jth *short rate process* $f^j = \{f^j(t),\, t \in [0,T]\}$ can be, for instance, a foreign interest rate or a dividend rate. The jth savings account measures accumulated income or loss generated by the jth asset in units of the jth asset, $j \in \{0,1,\dots,d\}$.

The i,jth *exchange price* $X^{i,j}(t)$ is the price of one unit of the jth asset at time t measured in units of the ith asset. The jth *savings account price measured in units of the ith primary asset*, denoted by $S^{i,j}(t)$ at time t, is given by

$$S^{i,j}(t) = X^{i,j}(t)\, B^j(t) \tag{2.2}$$

for $t \in [0,T]$ and $i,j \in \{0,1,\dots,d\}$.

We assume that there exists a growth optimal portfolio (GOP) which is a strictly positive, $\underline{\mathcal{A}}$-adapted, self-financing portfolio that when used as numeraire makes each benchmarked price process an $(\underline{\mathcal{A}}, P)$-local martingale. It then follows by definition that such a market is *locally arbitrage free*, see Jamshidian (1997) and Platen (2001a, 2001b). Note that in situations where the risk neutral pricing methodology can be applied, the GOP can be shown to be the inverse of the *state price deflator* or *state price density*, see Duffie (1996), Karatzas & Shreve (1998) and Platen (2001a, 2001b). However, we do not impose the somewhat restrictive assumption that an *equivalent risk neutral martingale measure* exists. This means the corresponding Radon-Nikodym derivative process is only assumed to be an $(\underline{\mathcal{A}}, P)$-local martingale and thus, in general, cannot be used to construct a standard equivalent risk neutral pricing measure, see Karatzas & Shreve (1998).

Let $D^j = \{D^j(t),\, t \in [0,T]\}$ denote the jth *denomination* of the GOP, when it is measured in units of the jth primary asset, $j \in \{0,1,\dots,d\}$. More precisely, we assume that $D^j(t)$ has the representation

$$D^j(t) = \sum_{\ell=0}^{d} \underline{\delta}^{\ell}(t)\, S^{j,\ell}(t) \tag{2.3}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$. Here $\underline{\delta}^{\ell}(t)$ denotes the number of units of the ℓth savings account held at time t in the GOP.

To be more specific and to demonstrate capabilities of the benchmark pricing methodology, a multi-factor diffusion model, the MMM, is now introduced. In the stylized version of the MMM considered here, the jth denomination $D^j(t)$ of the GOP at time t is specified as a transformation of the form

$$D^j(t) = (Y^j(t))^{q_j}\,\xi^j(t), \tag{2.4}$$

with the jth *average GOP*

$$\xi^j(t) = \xi^j(0)\,\exp\left\{\int_0^t \eta^j(s)\,ds\right\} \tag{2.5}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$. The deterministic jth *growth rate* $\eta^j = \{\eta^j(t),\, t \in [0,T]\}$ governs, according to (2.5), the jth average GOP. The jth *exponent*

$$q_j \in (0,\infty) \tag{2.6}$$

is constant. The jth *square root process* $Y^j = \{Y^j(t),\, t \in [0,T]\}$ in (2.4) is characterized by the stochastic differential equation (SDE)

$$dY^j(t) = \frac{\nu^j}{4}\,\varphi^j(t)\,(1 - Y^j(t))\,dt + \sum_{k=1}^{d} \gamma^{j,k}(t)\,\sqrt{Y^j(t)}\,dW^k(t) \tag{2.7}$$

with jth *diffusion* or *scaling parameter*

$$\varphi^j(t) = \sum_{k=1}^{d} (\gamma^{j,k}(t))^2 \tag{2.8}$$

for $t \in [0,T]$ and initial value $Y^j(0) > 0$, $j \in \{0,1,\dots,d\}$. Here $W^1,\dots,W^d$ are independent standard Wiener processes under the historical probability measure P. The jth *dimension* $\nu^j \in (2,\infty)$ is assumed to be constant and the j,kth *volatility parameter* $\gamma^{j,k} : [0,T] \to (-\infty,\infty)$ a deterministic function of time for $j \in \{0,1,\dots,d\}$, $k \in \{1,2,\dots,d\}$. Transformations other than (2.4) can also be used. However, for simplicity, in this paper we use only the simple transformation given in (2.4).

The square root process Y^j fluctuates around its reference level of one. The diffusion parameter φ^j controls the time scale of its evolution. The jth dimension ν^j provides a measure of the magnitude of extreme fluctuations. These extreme fluctuations are less likely for larger values of ν^j. The SDE (2.7) has a unique strong solution with an explicitly known transition density. Since $\nu^j > 2$ the process Y^j remains strictly positive w.p.1, see Karatzas & Shreve (1988).

2.2 Asset Price Dynamics

The jth *benchmarked savings account process* $\hat{S}^j = \{\hat{S}^j(t),\, t \in [0,T]\}$ is defined by the ratio

$$\hat{S}^j(t) = \frac{B^j(t)}{D^j(t)} \tag{2.9}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$.

By application of the Itô formula, using (2.9), (2.1), (2.4) and (2.7), the SDE for the jth benchmarked savings account is given by

$$\begin{aligned} d\hat{S}^j(t) &= \hat{S}^j(t)\left[f^j(t) - \eta^j(t) + q_j\,\frac{\nu^j}{4}\,\varphi^j(t) + \frac{\varphi^j(t)\,q_j}{Y^j(t)}\left(\frac{q_j+1}{2} - \frac{\nu^j}{4}\right)\right] dt \\ &\quad - \hat{S}^j(t)\sum_{k=1}^{d}\sigma^{j,k}(t)\,dW^k(t) \end{aligned} \tag{2.10}$$

for $t \in [0,T]$ with initial value $\hat{S}^j(0) = \frac{1}{D^j(0)}$. Here the j,kth *volatility* of the jth benchmarked savings account has the form

$$\sigma^{j,k}(t) = \frac{q_j\,\gamma^{j,k}(t)}{\sqrt{Y^j(t)}} \tag{2.11}$$

for $j \in \{0,1,\ldots,d\}$, $k \in \{1,2,\ldots,d\}$ and $t \in [0,T]$. To avoid redundant assets it is assumed that the volatility matrix $v(t) = [v^{k,i}(t)]_{k,i=0}^{d}$ with

$$v^{k,i}(t) = \begin{cases} 1 & \text{for} \quad k = 0 \\ \sigma^{i,k}(t) & \text{for} \quad k \in \{1,2,\ldots,d\} \end{cases} \tag{2.12}$$

is for all $t \in [0,T]$ *invertible.*

Since all benchmarked savings account processes are $(\underline{\mathcal{A}}, P)$-local martingales the drift term in the SDE (2.10) is zero. Therefore, the jth short rate is given by

$$f^j(t) = \eta^j(t) + q_j\,\varphi^j(t)\left\{\frac{1}{Y^j(t)}\left[\frac{\nu^j}{4} - \frac{q_j+1}{2}\right] - \frac{\nu^j}{4}\right\} \tag{2.13}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$, see Platen (2000b). From (2.9), (2.10) and (2.1), by application of the Itô formula, the jth denomination of the GOP satisfies the SDE

$$\begin{aligned} dD^j(t) &= d\left(\frac{B^j(t)}{\hat{S}^j(t)}\right) \\ &= D^j(t)\left[\left(f^j(t) + \sum_{k=1}^{d}(\sigma^{j,k}(t))^2\right) dt + \sum_{k=1}^{d}\sigma^{j,k}(t)\,dW^k(t)\right] \end{aligned} \tag{2.14}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$. The i,jth exchange price can be expressed by the ratio

$$X^{i,j}(t) = \frac{D^i(t)}{D^j(t)}, \tag{2.15}$$

for $t \in [0,T]$ and $i,j \in \{0,1,\dots,d\}$. Therefore, by the Itô formula together with (2.14), (2.2) and (2.1) the jth savings account, when measured in units of the ith asset, is governed by the SDE

$$dS^{i,j}(t) = S^{i,j}(t) \left[f^i(t)\,dt + \sum_{k=1}^{d} (\sigma^{i,k}(t) - \sigma^{j,k}(t)) \left\{ \sigma^{i,k}(t)\,dt + dW^k(t) \right\} \right] \tag{2.16}$$

for $t \in [0,T]$ with $S^{i,j}(0) = X^{i,j}(0)$ and $i,j \in \{0,1,\dots,d\}$. For $i = 0$, equation (2.16) describes the dynamics of the jth savings account expressed in units of the domestic currency. It can be observed from this SDE that the benchmark approach identifies a specific form for the risk premium.

Note that by (2.13), for

$$\nu^j = 2(q_j + 1) \tag{2.17}$$

a deterministic jth short rate of the form

$$f^j(t) = \eta^j(t) - \frac{1}{2}\, q_j\, (q_j + 1)\, \varphi^j(t) \tag{2.18}$$

is obtained for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$.

In the special case $\nu^j = 3$, $q_j = 0.5$ the benchmarked jth savings account $\hat{S}^j$ is the inverse of a three-dimensional Bessel process. This process is known to be a *strict* $(\underline{\mathcal{A}}, P)$*-local martingale*, see Protter (1990). In the standard risk neutral pricing methodology the process $\Lambda = \{\Lambda(t) = \frac{\hat{S}^0(t)}{\hat{S}^0(0)},\ t \in [0,T]\}$ would be the Radon-Nikodym derivative for the risk neutral measure. However, this process is not an $(\underline{\mathcal{A}}, P)$-martingale. Thus the standard risk neutral approach is not applicable in this case.

3 Pricing of Derivatives

We now use the benchmark pricing methodology, proposed in Platen (2001b), to price standard derivatives. As previously mentioned, this approach has the advantage that it can be applied in certain cases where the well-known risk neutral methodology fails.

Let $H^0_{\bar{T}} = H^0_{\bar{T}}(Y^0(\bar{T}), \dots, Y^d(\bar{T})) \in [0,\infty)$ be a nonnegative European payoff at the maturity date $\bar{T} \in (0,T]$, measured in units of the domestic currency. The corresponding *benchmarked payoff*

$$H_{\bar{T}} = H_{\bar{T}}(Y^0(\bar{T}), \ldots, Y^d(\bar{T})) = \frac{H^0_{\bar{T}}}{D^0(\bar{T})} \tag{3.1}$$

is assumed to be integrable, that is

$$E(|H_{\bar{T}}|) < \infty. \tag{3.2}$$

As shown in Platen (2001b), one obtains for certain payoffs and parameter choices a class of *benchmarked pricing functions* that allow perfect hedging via corresponding self-financing hedging strategies. In particular, for a given benchmarked payoff $H_{\bar{T}}$, such a benchmarked pricing function $u : [0, \bar{T}] \times (0, \infty)^{d+1} \to [0, \infty)$ satisfies the partial differential equation (PDE)

$$\left(\frac{\partial}{\partial t} + \sum_{\ell=0}^{d} \frac{\nu^\ell}{4} \varphi^\ell(t) \left(1 - Y^\ell(t)\right) \frac{\partial}{\partial Y^\ell} \right.$$
$$\left. + \frac{1}{2} \sum_{\ell,r=0}^{d} \sum_{k=1}^{d} \gamma^{\ell,k}(t) \gamma^{r,k}(t) \sqrt{Y^\ell(t)\, Y^r(t)} \frac{\partial^2}{\partial Y^\ell \, \partial Y^r} \right) u(t, Y^0, \ldots, Y^d) = 0 \tag{3.3}$$

for $(t, Y^0, \ldots, Y^d) \in [0, \bar{T}) \times (0, \infty)^{d+1}$ with terminal condition

$$u(\bar{T}, Y^0, \ldots, Y^d) = H_{\bar{T}}(Y^0, \ldots, Y^d) \tag{3.4}$$

for $(Y^0, \ldots, Y^d) \in (0, \infty)^{d+1}$. Note that in cases where several distinct solutions for this PDE exist then several distinct hedging strategies also exist. In Section 5 two specific examples with different solutions for the PDE (3.3) - (3.4) are given.

Now, let us introduce for a benchmarked pricing function u the vector $c_u(t) = (c^0_u(t), \ldots, c^d_u(t))^\top$ with $c^0_u(t) = 1$ and

$$c^k_u(t) = -\sum_{\ell=0}^{d} \gamma^{\ell,k}(t) \sqrt{Y^\ell(t)} \frac{\partial \log(u(t, Y^0(t), \ldots, Y^d(t)))}{\partial Y^\ell} \tag{3.5}$$

for $t \in [0, \bar{T}]$ and $k \in \{1, 2, \ldots, d\}$. It has been shown in Platen (2001b) that for a benchmarked pricing function u the corresponding self-financing hedging portfolio has the vector of *proportions*

$$(\pi^0_u(t), \ldots, \pi^d_u(t))^\top = v^{-1}(t)\, c_u(t) \tag{3.6}$$

for $t \in [0, \bar{T}]$. The quantity $\pi^j_u(t)$ is the proportion of the value of the jth savings account held in the hedging portfolio, that is

$$\pi^j_u(t) = \frac{\delta^j(t)\, \hat{S}^j(t)}{u(t, Y^0(t), \ldots, Y^d(t))} \tag{3.7}$$

for $t \in [0, \bar{T}]$, $j \in \{0, 1, \ldots, d\}$. Then $\delta^j(t)$ denotes the number of units of the jth savings account held in this hedging portfolio. The resulting hedging strategy allows a perfect hedge as long as the benchmarked pricing function u satisfies the PDE (3.3) and (3.4). This point will be clarified in Section 5.

4 Basic Index Derivatives

4.1 Dynamics of the GOP

In this section we consider the case, where $\nu^j = 2(q_j + 1)$, see (2.17). This means we assume a deterministic short rate $f^0(t)$, as can be seen from (2.18).

According to (2.14), (2.8) and (2.11) the SDE for the domestic GOP is given by

$$dD^0(t) = D^0(t)\left[\left(f^0(t) + \frac{(q_0)^2\,\varphi^0(t)}{Y^0(t)}\right)dt + \frac{q_0}{\sqrt{Y^0(t)}}\sqrt{\varphi^0(t)}\,d\bar{W}^0(t)\right], \qquad (4.1)$$

where

$$d\bar{W}^0(t) = \frac{1}{\sqrt{\varphi^0(t)}}\sum_{k=1}^{d}\gamma^{0,k}(t)\,dW^k(t) \qquad (4.2)$$

for $t \in [0, T]$. Since

$$d\langle\bar{W}^0\rangle_t = \frac{\sum_{k=1}^{d}(\gamma^{0,k}(t))^2}{\varphi^0(t)}\,dt = dt \qquad (4.3)$$

for $t \in [0, T]$, see (4.2) and (2.8), it follows by Lévy's theorem that $\bar{W}^0$ is a standard Wiener process under the measure P. Combining (2.7) and (4.2) we obtain

$$dY^0(t) = \frac{\nu^0}{4}\,\varphi^0(t)\,(1 - Y^0(t))\,dt + \sqrt{\varphi^0(t)\,Y^0(t)}\,d\bar{W}^0(t) \qquad (4.4)$$

for $t \in [0, T]$.

The value $D^0(t)$, of the GOP when dominated in units of the domestic currency at time t, is also given by the sum (2.3). This representation means that the GOP can be interpreted as a large diversified portfolio and thus as a cumulative market index. This fact together with (4.4), shows that the uncertainty in the pricing and hedging of index derivatives depends only on the Wiener process $\bar{W}^0$. In the following subsections we consider basic derivatives on the GOP.

4.2 Zero Coupon Bond

A surprisingly simple index derivative is formed by a zero coupon bond that pays one unit of the domestic currency at the maturity date $\bar{T}$. Its benchmarked payoff at time $\bar{T}$ is $H_{\bar{T}} = \frac{1}{D^0(\bar{T})}$. We denote by $P^0_*(t, \bar{T})$ the traded price, in domestic currency, of this zero coupon bond at time $t \in [0, \bar{T}]$. Since the short rate process f^0 is, in our example, deterministic we have

$$P^0_*(t, \bar{T}) = \frac{B^0(t)}{B^0(\bar{T})} \qquad (4.5)$$

for $t \in [0, \bar{T}]$. The corresponding *benchmarked traded zero coupon bond price* $\hat{P}^0_*(t, \bar{T})$ satisfies the relation

$$\begin{aligned}
\hat{P}^0_*(t,\bar{T}) &= \frac{1}{D^0(t)}\,\frac{B^0(t)}{B^0(\bar{T})} \\
&= \frac{\hat{S}^0(t)}{B^0(\bar{T})} \\
&= \frac{\exp\left\{-\int_0^t \eta^0(s)\,ds - \int_t^{\bar{T}} f^0(s)\,ds\right\}}{(Y^0(t))^{q_0}\,\xi^0(0)} \\
&= u_*(t, Y^0(t)) \qquad (4.6)
\end{aligned}$$

for $t \in [0, \bar{T}]$ and some function $u_* : [0, \bar{T}] \times (0, \infty) \to [0, \infty)$. It is straightforward to check that the function u_* solves the PDE (3.3) - (3.4). By the Itô formula together with (4.6), (4.1) and (2.1) the SDE for $\hat{P}^0_*(\cdot, \bar{T})$ takes the form

$$d\hat{P}^0_*(t,\bar{T}) = -\hat{P}^0_*(t,\bar{T})\,q_0\sqrt{\frac{\varphi^0(t)}{Y^0(t)}}\,d\bar{W}^0(t) \qquad (4.7)$$

for $t \in [0, \bar{T}]$. It can be shown that

$$P\left(\int_0^{\bar{T}} \left(u_*(s, Y^0(s))\,q_0\right)^2 \frac{\varphi^0(s)}{Y^0(s)}\,ds < \infty\right) = 1. \qquad (4.8)$$

Consequently, the process $\hat{P}^0_*(\cdot, \bar{T})$ is an $(\underline{\mathcal{A}}, P)$-local martingale and since it is nonnegative, it is an $(\underline{\mathcal{A}}, P)$-supermartingale, see Karatzas & Shreve (1988). This proves the inequalities

$$\hat{P}^0_*(t,\bar{T}) \geq E\left(\hat{P}^0_*(s,\bar{T}) \,\middle|\, \mathcal{A}_t\right) \geq E\left(\frac{1}{D^0(\bar{T})} \,\middle|\, \mathcal{A}_t\right) \qquad (4.9)$$

for $t \in [0, \bar{T}]$ and $s \in [t, \bar{T}]$.

These inequalities lead us to consider another benchmarked price $\hat{P}^0_m(t, \bar{T})$, called the *benchmarked arbitrage free bond price*, given by the conditional expectation

$$\begin{aligned}
\hat{P}^0_m(t,\bar{T}) &= E\left(\frac{1}{D^0(\bar{T})} \,\middle|\, \mathcal{A}_t\right) \\
&= E\left(\frac{1}{(Y^0(\bar{T}))^{q_0}\,\xi^0(\bar{T})} \,\middle|\, \mathcal{A}_t\right) \\
&= u_m(t, Y^0(t)) \qquad (4.10)
\end{aligned}$$

for $t \in [0, \bar{T}]$ and some function $u_m : [0, \bar{T}] \times (0, \infty) \to [0, \infty)$. The function u_m also satisfies the PDE (3.3) - (3.4) and can be used to perfectly hedge the payoff

$H_{\bar{T}} = \frac{1}{D^0(\bar{T})}$. The conditional expectation appearing in (4.10) can be explicitly computed using the well-known transition density of the square root process, see Revuz & Yor (1999). In domestic currency, this alternative bond price is then given by the formula

$$P_m^0(t,\bar{T}) = D^0(t)\, \hat{P}_m^0(t,\bar{T}) \tag{4.11}$$

for $t \in [0,\bar{T}]$. For our case, with a deterministic short rate, the two bond prices $P_*(t,\bar{T})$ and $P_m(t,\bar{T})$ are different for $t \in [0,\bar{T})$. We call the difference

$$A^0(t,\bar{T}) = P_*^0(t,\bar{T}) - P_m^0(t,\bar{T}) \tag{4.12}$$

the corresponding *arbitrage amount*, which is expressed here in units of the domestic currency.

Figure 4.1 shows the arbitrage free bond price $P_m^0(t,\bar{T})$ in domestic currency as a function of time t and the initial GOP value $D^0(0)$. For this and subsequent plots displayed in this section the default parameter values $\bar{T} = 10$, $\nu^0 = 3$, $\varphi^0 = 0.16$, $\xi^0(0) = 1$, $\eta^0(t) = 0.11$, $t \in [0,\bar{T}]$ and $q_0 = 0.5$ were used. The corresponding domestic short rate is therefore, by (2.18), $f^0(t) = 0.05$.

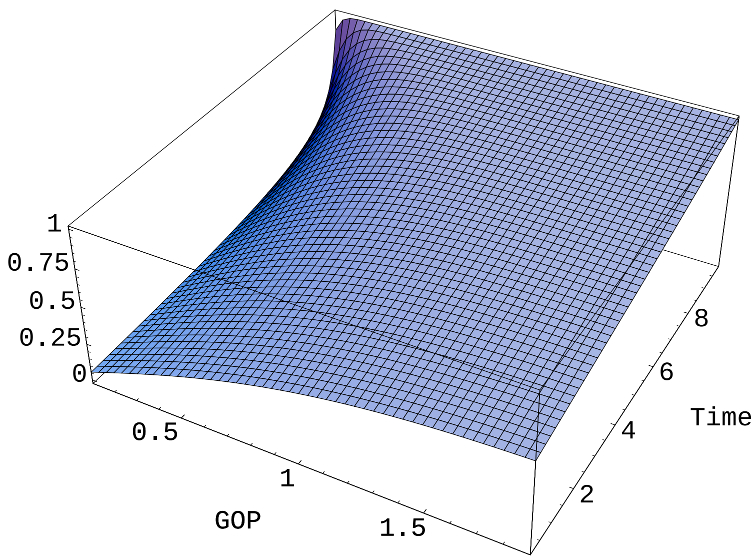

Figure 4.1: Arbitrage free bond price $P_m^0(t,\bar{T})$ as a function of $D^0(t)$ and time t.

Note from Figure 4.1 that the arbitrage free bond price $P_m^0(t,\bar{T})$ depends both on the initial GOP value $D^0(t)$ and time t. However, the traded bond price $P_*^0(t,\bar{T})$ depends only on the time parameter. This indicates together with the inequalities (4.9) that the arbitrage amount $A^0(t,\bar{T})$ is nonnegative. This is also visible in Figure 4.2, which displays the arbitrage amount as a function of the actual GOP value $D^0(t)$ and time t. Note that for small values of the GOP and the parameter values used above, some significant arbitrage amounts are generated.

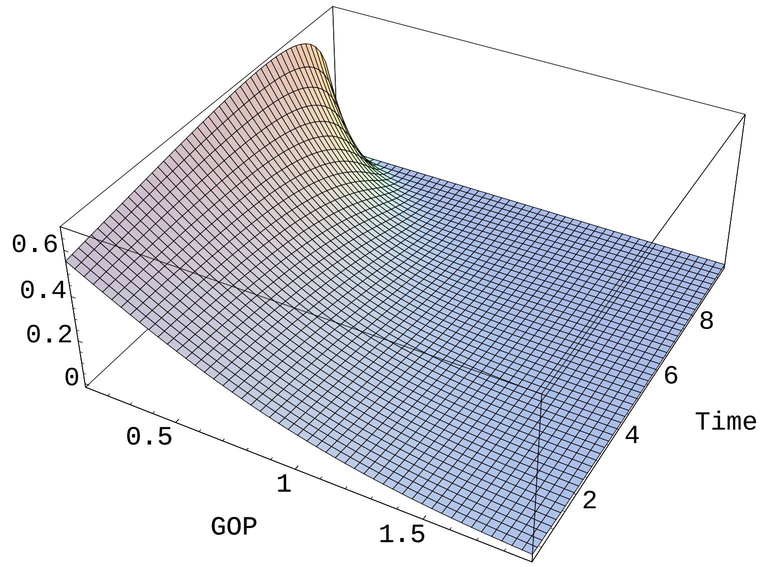

Figure 4.2: Arbitrage amount for zero coupon bond.

4.3 European Call Option

We now consider the case of a European call option on the GOP with maturity date $\bar{T}$. Here the payoff, expressed in units of the domestic currency, is

$$H^0_{\bar{T}} = (D^0(\bar{T}) - K)^+,$$

where K is the strike price. Consequently, the benchmarked payoff

$$H_{\bar{T}} = \left(1 - \frac{K}{D^0(\bar{T})}\right)^+ \tag{4.13}$$

is bounded. Since the above benchmarked payoff is bounded it follows that the *benchmarked option price* $\hat{c}^0_{\bar{T},K}(t, Y^0(t))$ is given by the conditional expectation

$$\begin{aligned} \hat{c}^0_{\bar{T},K}(t, Y^0(t)) &= E\left(\left(1 - \frac{K}{D^0(\bar{T})}\right)^+ \middle| \mathcal{A}_t\right) \\ &= u_c(t, Y^0(t)) \end{aligned} \tag{4.14}$$

for $t \in [0, \bar{T}]$ and some function $u_c : [0, \bar{T}] \times (0, \infty) \to [0, \infty)$, see Platen (2001b). It can be shown that the function u_c satisfies the PDE (3.3) - (3.4). The corresponding price $c^0_{\bar{T},K}(t, Z^0(t))$ for this contingent claim, when denominated in units of the domestic currency, is given by

$$c^0_{\bar{T},K}(t, Y^0(t)) = D^0(t)\, \hat{c}^0_{\bar{T},K}(t, Y^0(t)) \tag{4.15}$$

for $t \in [0, \bar{T}]$. It can also be shown that the solution of the PDE (3.3) - (3.4) with terminal condition (4.13) is unique within the class of bounded solutions for this PDE. Consequently, u_c is the unique solution for the price of this European call.

The implied volatility term structure is often used to determine the deviation of option prices from those of corresponding Black-Scholes option prices. In Figure 4.3, the implied volatility term structure, obtained from (4.15), is displayed using different values of strike K and time t.

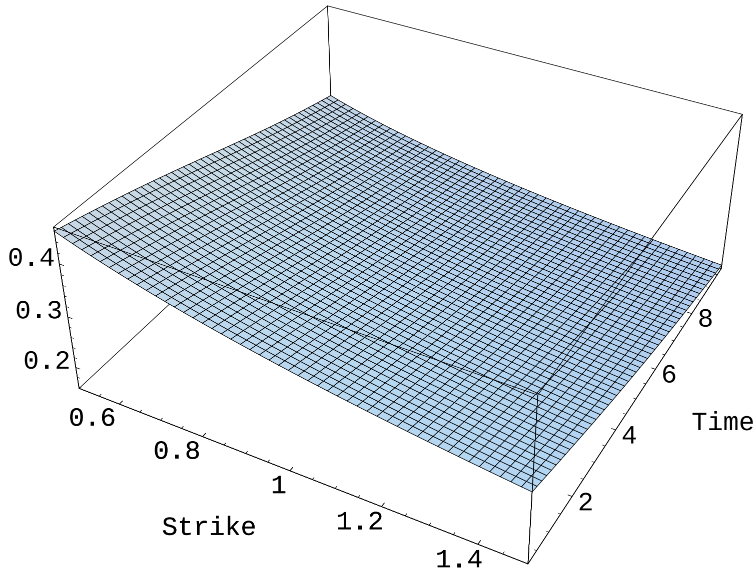

Figure 4.3: Implied volatilities for European calls.

The implied volatility surface in Figure 4.3 shows a negative skew, a feature which is typically observed in many markets, see, for instance, Derman (1999). It naturally arises in the MMM without the requirement of introducing an external stochastic volatility process.

In the case of a European put with strike K and maturity date $\bar{T} \in [0, T]$, the corresponding traded put price $p^0_{*,\bar{T},K}(t, Y^0(t))$ can be obtained from the put-call parity relation

$$p^0_{*,\bar{T},K}(t, Y^0(t)) = c^0_{\bar{T},K}(t, Y^0(t)) - D^0(t) + K\, P^0_*(t, \bar{T}) \tag{4.16}$$

for $t \in [0, \bar{T}]$. This value can be determined since the prices for traded bonds and European calls have been previously computed. As seen from this relation, we need to use the price of traded bonds given in (4.5). For the resulting traded put prices the implied volatility would be the same as shown in Figure 4.3. However, if we use the benchmarked arbitrage free bond price, see (4.11), to obtain the benchmarked put price via the corresponding put-call parity relation, then the resulting benchmarked pricing function would satisfy the PDE (3.3) - (3.4). Therefore, this price process can also be used to perfectly hedge a European put.

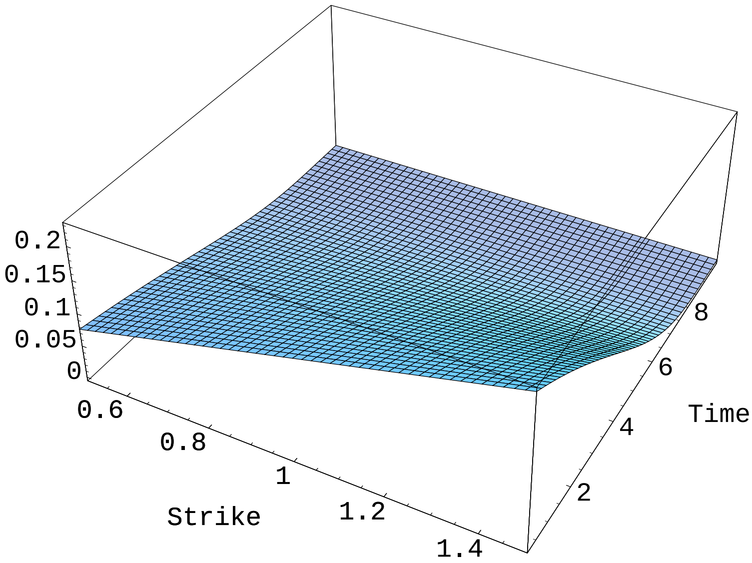

Figure 4.4: Arbitrage amount from traded and arbitrage free put prices.

Figure 4.4 shows the difference between the put price obtained from the traded bond, see (4.5), and that derived from the arbitrage free bond price (4.11). Since perfect hedging is associated with both price processes this difference constitutes an arbitrage amount that exists for this class of securities. For the arbitrage free put price the corresponding implied volatility term structure is displayed in Figure 4.5. These implied volatilities are clearly different from those shown in Figure 4.3.

5 Hedging Prescriptions

As previously explained, the two distinct bond price processes are both associated with distinct perfect hedging prescriptions for the corresponding derivative security. In this section, we describe in more detail how the corresponding payoff for a zero coupon bond can be perfectly hedged. Similar arguments apply for the traded and arbitrage free put prices, which we omit here.

For the traded bond price given in (4.5) we form a hedging portfolio with $\delta^0_*(t)$ units of the domestic savings account and $\delta^1_*(t)$ units held in the GOP. With an appropriate choice for the hedge ratios $\delta^0_*(t)$ and $\delta^1_*(t)$, specified below, the value of this hedging portfolio measured in units of the domestic currency can be made equal to $P^0_*(t, \bar{T})$, that is

$$P^0_*(t, \bar{T}) = \delta^0_*(t)\, B^0(t) + \delta^1_*(t)\, D^0(t) \tag{5.1}$$

for $t \in [0, \bar{T}]$. Therefore, the benchmarked value of this hedging portfolio is given

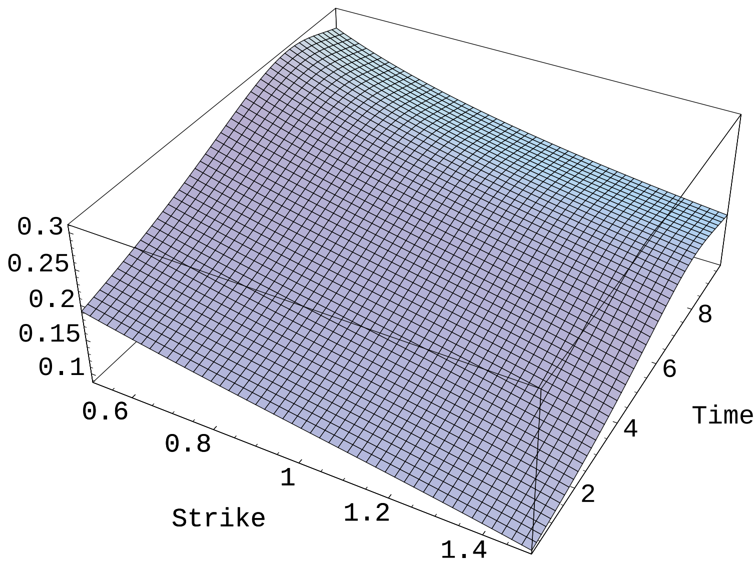

Figure 4.5: Implied volatilities for arbitrage free put prices.

by

$$\hat{P}^0_*(t,\bar{T}) = \delta^0_*(t)\,\hat{S}^0(t) + \delta^1_*(t) \tag{5.2}$$

for $t \in [0,\bar{T}]$. In addition, applying the Itô formula to the function u_* given in (4.6), together with the PDE (3.3) - (3.4) and (2.10) - (2.11), we see that

$$\begin{aligned} \hat{P}^0_*(t,\bar{T}) &= u_*(t, Y^0(t)) \\ &= u_*(0, Y^0(0)) + \int_0^t \frac{\partial u_*(s, Y^0(s))}{\partial Y^0}\,\gamma^0\,\sqrt{Y^0(s)}\,d\bar{W}^0(s) \\ &= u_*(0, Y^0(0)) + \int_0^t \zeta^0_*(s)\,d\hat{S}^0(s) \end{aligned} \tag{5.3}$$

with

$$\zeta^0_*(t) = \frac{1}{B^0(\bar{T})} \tag{5.4}$$

for $t \in [0,\bar{T}]$. Now, we define the *benchmarked costs* $\hat{C}^0_*(t)$ of maintaining this hedging portfolio up to time $t \in [0,\bar{T}]$ as the benchmarked value of the hedging portfolio minus the *benchmarked gains from trade*, that is

$$\hat{C}^0_*(t) = \hat{P}^0_*(t,\bar{T}) - \int_0^t \delta^0_*(s)\,d\hat{S}^0(s) \tag{5.5}$$

for $t \in [0,\bar{T}]$. Choosing the hedge ratio $\delta^0_*(t) = \zeta^0_*(t)$ so that

$$\delta^1_*(t) = \hat{P}^0_*(t,\bar{T}) - \delta^0_*(t)\,\hat{S}^0(t)$$

for $t \in [0, \bar{T}]$ and combining (5.5) with (5.3) yields

$$\begin{aligned} \hat{C}^0_*(t) &= \hat{P}^0_*(0, \bar{T}) \\ &= u_*(0, Y^0(0)) \end{aligned} \tag{5.6}$$

for all $t \in [0, \bar{T}]$. This shows us that the benchmarked costs for this hedging strategy equal the initial benchmarked price of the traded zero coupon bond and are therefore constant.

Let us denote by $\bar{P}^0_*(t, \bar{T})$ the value of the above hedging portfolio, discounted by the domestic savings account $B^0(t)$, which is given by the expression

$$\begin{aligned} \bar{P}^0_*(t, \bar{T}) &= \frac{P^0_*(t, \bar{T})}{B^0(t)} \\ &= \delta^0_*(t) + \delta^1_*(t) \, \frac{D^0(t)}{B^0(t)} \end{aligned} \tag{5.7}$$

for $t \in [0, \bar{T}]$. We refer to this expression as the savings account discounted portfolio value. The corresponding *savings account discounted costs* $\bar{C}^0_*(t)$ are given by

$$\bar{C}^0_*(t) = \bar{P}^0_*(t, \bar{T}) - \int_0^t \delta^1_*(s) \, d\left(\frac{D^0(s)}{B^0(s)}\right) \tag{5.8}$$

for $t \in [0, \bar{T}]$. It can be shown, using similar arguments to those outlined above, that

$$\bar{C}^0_*(t) = \bar{P}^0_*(0, \bar{T}) \tag{5.9}$$

for $t \in [0, \bar{T}]$. This relation is therefore the analogue of (5.6) expressed in discounted terms.

In the case of the benchmarked arbitrage free bond price similar formulas can be derived. In particular, we obtain the following expressions for the arbitrage free bond price $P^0_m(t, \bar{T})$, the benchmarked arbitrage free bond price $\hat{P}^0_m(t, \bar{T})$ and *benchmarked arbitrage free costs* $\hat{C}^0_m(t)$:

$$P^0_m(t, \bar{T}) = \delta^0_m(t) \, B^0(t) + \delta^1_m(t) \, D^0(t), \tag{5.10}$$

$$\begin{aligned} \hat{P}^0_m(t, \bar{T}) &= \delta^0_m(t) \, \hat{S}^0(t) + \delta^1_m(t) \\ &= u_m(0, Y^0(0)) + \int_0^t \zeta^0_m(s) \, d\hat{S}^0(s) \end{aligned} \tag{5.11}$$

with

$$\zeta^0_m(t) = -\frac{\zeta^0_m(0)}{q_0} \, (Y^0(t))^{q_0+1} \, \exp\left\{\frac{1}{2} \, q_0 \, (q_0+1) \int_0^t \varphi^0(s) \, ds\right\} \, \frac{\partial u_m(s, Y^0(s))}{\partial Y^0} \tag{5.12}$$

and

$$\begin{aligned}\hat{C}^0_m(t) &= \hat{P}^0_m(t,\bar{T}) - \int_0^t \delta^0_m(s)\, d\hat{S}^0(s) \\ &= \hat{P}^0_m(0,\bar{T}) \end{aligned} \tag{5.13}$$

with $\delta^0_m(t) = \zeta^0_m(t)$ for $t \in [0, \bar{T}]$.

Similar formulas to those given in (5.7) - (5.9) can also be obtained for the corresponding savings account discounted arbitrage free bond price and savings account discounted arbitrage free costs.

6 Hedging Simulation

In this section we consider some hedge simulations, which show that for certain parameter choices for the MMM a form of arbitrage exists.

Again, the case $\nu^0 = 3$ and $q_0 = 0.5$ is considered with the other MMM parameters as specified in Section 4.2. The simulated benchmarked prices for the traded and arbitrage free bond are displayed in Figure 6.1 for a particular sample path of the underlying square root process Y^0. Inspection of this figure indicates that the arbitrage free benchmarked price is always less than or equal to the corresponding benchmarked traded price. Also the benchmarked arbitrage amount $\hat{A}^0(0,\bar{T})$ is nonnegative. It fluctuates randomly, but eventually tends to zero as we proceed along the trajectory. This is due to the supermartingale inequality (4.9) and the fact that each benchmarked price process equals the benchmarked payoff $H_{\bar{T}} = \frac{1}{D^0(\bar{T})}$ at maturity.

The corresponding hedge ratios δ^0_* and δ^0_m for the traded and arbitrage free price processes, respectively, are shown in Figure 6.2 for the same sample path. As seen in (5.4) the hedge ratio δ^0_* is constant. The difference $\delta^0_* - \delta^0_m$ is nonnegative but also eventually tends to zero as the maturity date $\bar{T}$ approaches.

To verify the fact that both benchmarked price processes are associated with perfect hedging prescriptions we show in Figure 6.3 the benchmarked costs $\hat{C}^0_*$ and $\hat{C}^0_m$ for the same sample path. As can be seen from this figure both, benchmarked costs are nearly constant. These benchmarked costs were computed using five hundred discretization points corresponding to approximately weekly hedging for the ten year bond. Consequently, even for relatively coarse hedging, the benchmarked costs can be kept almost constant. For more frequent hedging the variability of these benchmarked costs can be further reduced.

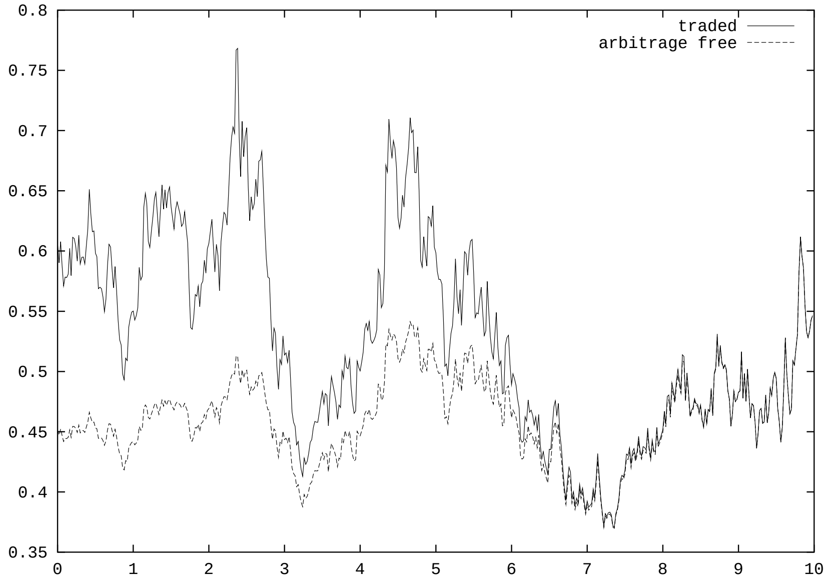

Figure 6.1: Simulated benchmarked traded and arbitrage free bond prices.

Conclusion

The above market model is complete and can be used to price and hedge index derivatives. The implied volatility term structure obtained for European calls and puts shows negative skews as is typically observed in real markets. As demonstrated in this paper the benchmark pricing methodology allows us to handle certain interesting cases that permit a form of arbitrage, which can be explicitly characterized. If one chooses the parameters so that the Radon-Nikodym derivative is a martingale, then the risk neutral methodology can be applied and is equivalent to the benchmark approach. The benchmark methodology can be extended to more complex derivatives and more general asset price dynamics.

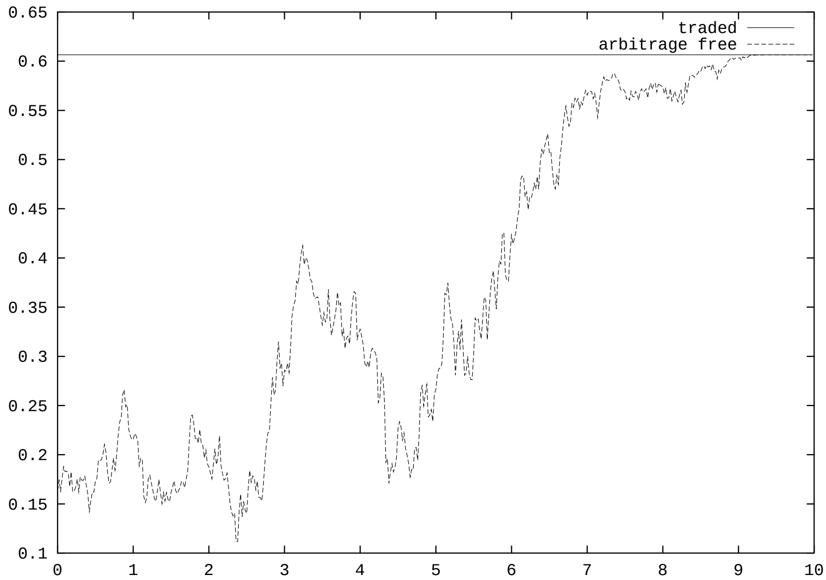

Figure 6.2: Hedge ratios δ_*^0 and δ_m^0 for bonds.

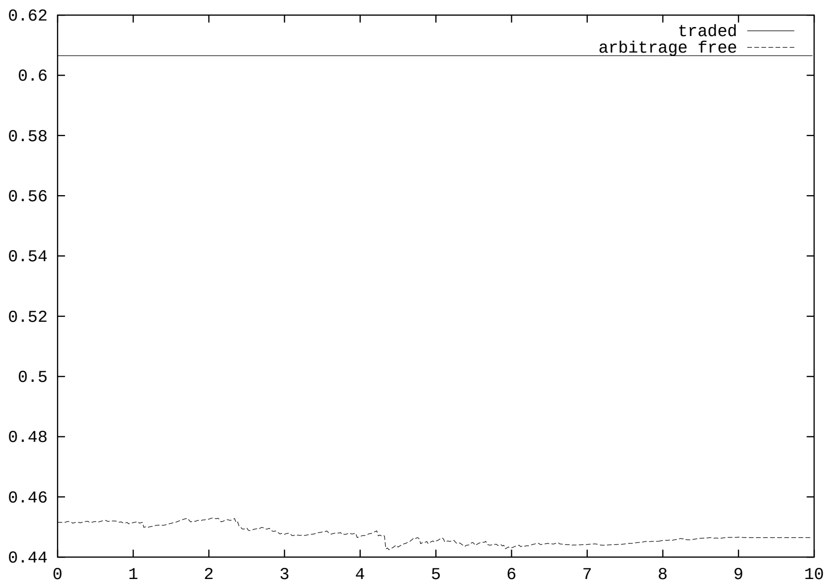

Figure 6.3: Benchmarked costs $\hat{C}_*^0$ and $\hat{C}_m^0$ for bonds.

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