The Least Cost Super Replicating Portfolio in the Boyle-Vorst Discrete-Time Option Pricing Model with Transactions Costs

Ken Palmer
Department of Finance
University of Melbourne, Victoria 3010, Australia
Tel. 61-3-83445318; fax 61-3-83446914; palmer@unimelb.edu.au

Abstract

Working in a binomial framework, Boyle and Vorst derived self-financing strategies perfectly replicating the final payoffs to long positions in European call and put options, assuming proportional transactions costs on trades in the stocks. The initial cost of such a strategy yields, by an arbitrage argument, an upper bound for the option price. A lower bound for the option price is obtained by replicating a short position.

It is known that no super replicating portfolio for positions in long calls and puts can have a lower cost than the replicating portfolio. However, even when a short call or put has a unique replicating portfolio, there may exist super replicating portfolios of lower cost when transactions costs are sufficiently large. The negative of the least possible cost of such a super replicating portfolio would be a lower bound for the call or put price. Now the cost of the replicating portfolio can easily be calculated by backward recursion. However, as there are possibly infinitely many super replicating portfolios, it is not immediately obvious how the least possible cost of a super replicating portfolio can be efficiently calculated. The aim of this paper to to describe progress towards the development of an efficient method to calculate this least cost.

Key Words: option pricing, transactions costs, discrete-time model, super replicating portfolios

JEL Classification: G13, C63

1. Introduction

The Black-Scholes model (conf. Black and Scholes (1973)) for option pricing assumes that markets are frictionless, in particular that there are no transactions costs. Since 1973 many authors have attempted to develop option pricing models incorporating transactions costs, the first major work being that of Leland (1985) who works in a continuous-time framework. Further work on option pricing with transactions costs in the continuous-time framework has been done by, for example, Hodges and Neuberger (1989), Davis, Panas and Zariphopoulou (1992), Grannan and Swindle (1996), Toft (1996). However, in this paper we work in the discrete framework.

Boyle and Vorst (1992) work in the framework of the binomial model and derive self-financing strategies perfectly replicating the final payoffs to long and short positions in call and put options, assuming proportional transactions costs on trades in the stock and no transactions costs on trades in the bonds. By the usual arbitrage argument, the cost of replicating a long position (that is, the initial value of the replicating portfolio) gives an upper bound for the option price whereas the negative of the cost of replicating a short position gives a lower bound for the option price.

In Palmer (2001) we clarified the conditions under which there is a unique replicating strategy in the Boyle-Vorst model for an arbitrary contingent claim. Actually, following Stettner (1997) and Rutkowski (1998), we worked in the framework of asymmetric proportional transactions costs, which includes not only the model of Boyle and Vorst but also the slightly different model of Bensaid, Lesne, Pages and Scheinkman (1992). In the one-period case we showed that under a very mild condition, replicating portfolios always exist and we determined their exact number. In particular, we determined when they are unique.

Even when the market is arbitrage-free and a given contingent claim has a unique replicating portfolio, there may exist super replicating portfolios of lower cost. However, Bensaid, Lesne, Pages and Scheinkman give conditions under which the cost of the replicating portfolio does not exceed the cost of any super replicating portfolio. These results were generalised by Stettner and Rutkowski to the case of asymmetric transactions costs. In Palmer (2001) we provided a further slight generalisation and gave what seems to be a simpler proof. These results have the consequence that there is no super replicating portfolio for long calls and puts of lower cost than the replicating portfolio. However, this is is not true for short calls and puts. Since the negative of the cost of the least cost super replicating portfolio for such a position is a lower bound for the call or put price, it is important to determine this least cost.

In this paper we determine the least cost super replicating portfolio in the one-period case for any contingent claim. Here it is assumed that the current stock price is S and that in the next period it either rises to Su or falls to Sd, where

$$0 < d < R < u,$$

R being the value of one unit of bond after one period. We also assume that buying one share incurs a transaction cost of λS and that selling one share incurs a transaction cost of μS, where

$$\lambda \geq 0, \quad 0 \leq \mu < 1.$$

A *contingent claim* consists of holdings (Δ_u, B_u) of Δ_u shares and B_u units of bonds in the upstate when the share price is Su and holdings (Δ_d, B_d) of Δ_d shares and B_d units of bonds in the downstate when the share price is Sd. A *super replicating portfolio* for this contingent claim consists of current holdings (Δ, B) of Δ shares and B bonds which can be rebalanced at the next period so as to "dominate" the contingent claim, that is, there exist $(\overline{\Delta}_u, \overline{B}_u)$ and $(\overline{\Delta}_d, \overline{B}_d)$ such that

$$\overline{\Delta}_u \geq \Delta_u, \quad \overline{B}_u \geq B_u,$$

$$\overline{\Delta}_d \geq \Delta_d, \quad \overline{B}_d \geq B_d$$

and

$$\begin{aligned} \Delta Su + BR &= \overline{\Delta}_u Su + \overline{B}_u + (\overline{\Delta}_u - \Delta) Su \overline{e}_u(\Delta), \\ \Delta Sd + BR &= \overline{\Delta}_d Sd + \overline{B}_d + (\overline{\Delta}_d - \Delta) Sd \overline{e}_d(\Delta), \end{aligned} \tag{1}$$

where

$$\overline{e}_u(\Delta) = \begin{cases} -\mu & \text{if } \Delta \geq \overline{\Delta}_u \\ \lambda & \text{if } \Delta < \overline{\Delta}_u \end{cases}, \quad \overline{e}_d(\Delta) = \begin{cases} -\mu & \text{if } \Delta \geq \overline{\Delta}_d \\ \lambda & \text{if } \Delta < \overline{\Delta}_d \end{cases}.$$

The cost of this super replicating portfolio is

$$\Delta S + B.$$

(Note that (Δ, B) is a *replicating portfolio* if $\overline{\Delta}_u = \Delta_u$, $\overline{B}_u = B_u$, $\overline{\Delta}_d = \Delta_d$, $\overline{B}_d = B_d$.) Without loss of generality we can assume that

$$\overline{\Delta}_u = \Delta_u, \quad \overline{\Delta}_d = \Delta_d$$

for if the equations (1) hold, the equations

$$\begin{aligned} \Delta Su + BR &= \Delta_u Su + \tilde{B}_u + (\Delta_u - \Delta) Su e_u(\Delta) \\ \Delta Sd + BR &= \Delta_d Sd + \tilde{B}_d + (\Delta_d - \Delta) Sd e_d(\Delta) \end{aligned}$$

also hold, where

$$e_u(\Delta) = \begin{cases} -\mu & \text{if } \Delta \geq \Delta_u \\ \lambda & \text{if } \Delta < \Delta_u \end{cases}, \quad e_d(\Delta) = \begin{cases} -\mu & \text{if } \Delta \geq \Delta_d \\ \lambda & \text{if } \Delta < \Delta_d \end{cases} \tag{2}$$

and it is easily verified that

$$\tilde{B}_u \geq \overline{B}_u, \quad \tilde{B}_d \geq \overline{B}_d.$$

We determine the least cost super replicating portfolio in two steps. In Theorem 1, we show that if the least cost super replicating portfolio is not a replicating portfolio, then there are just two other possibilities. Then in Theorem 2, we determine the situations in which each possibility arises.

2. The First Theorem.

In this section we consider a one-period model with contingent claim $\{(\Delta_u, B_u), (\Delta_d, B_d)\}$ and show that a least cost super replicating portfolio exists and it is either a replicating portfolio or a super replicating portfolio with holdings either $(\Delta_u, B_u/R)$ or $(\Delta_d, B_d/R)$.

Theorem 1. *Consider a one-period model with contingent claim (Δ_u, B_u) in the up state and (Δ_d, B_d) in the down state. Then a least cost super replicating portfolio exists and it is either a replicating portfolio or a super replicating portfolio with holdings either $(\Delta_u, B_u/R)$ or $(\Delta_d, B_d/R)$.*

Proof. We begin by fixing Δ, the share holdings, and we determine all bond holdings B such that (Δ, B) are the holdings in a super replicating portfolio for the contingent claim $\{(\Delta_u, B_u), (\Delta_d, B_d)\}$.

Suppose first that (Δ, B) are the holdings in a super replicating portfolio. Then there exist $\overline{B}_u$ and $\overline{B}_d$ such that

$$\overline{B}_u \geq B_u, \quad \overline{B}_d \geq B_d$$

and

$$\begin{aligned} \Delta Su + BR &= \Delta_u Su + \overline{B}_u + (\Delta_u - \Delta) Su e_u(\Delta), \\ \Delta Sd + BR &= \Delta_d Sd + \overline{B}_d + (\Delta_d - \Delta) Sd e_d(\Delta), \end{aligned}$$

where $e_u(\Delta)$ and $e_d(\Delta)$ are given in (2). It follows that

$$\overline{B}_u = BR + (\Delta - \Delta_u) Su(1 + e_u(\Delta)), \quad \overline{B}_d = BR + (\Delta - \Delta_d) Su(1 + e_d(\Delta)) \tag{3}$$

and that B satisfies

$$BR \geq (\Delta_u - \Delta) Su(1 + e_u(\Delta)) + B_u, \quad BR \geq (\Delta_d - \Delta) Su(1 + e_d(\Delta)) + B_d. \tag{4}$$

Conversely, if we choose B satisfying (4) and define $\overline{B}_u$ $\overline{B}_d$ by (3), then it follows by reversing the argument that (Δ, B) are the holdings in a super replicating portfolio.

Hence, for the chosen Δ, (Δ, B) are the holdings in a super replicating portfolio for the contingent claim $\{(\Delta_u, B_u), (\Delta_d, B_d)\}$ if and only if B satisfies (2).

It follows that the minimum possible value for $\Delta S + B$ with this Δ is

$$h(\Delta) = \Delta S + B(\Delta),$$

where

$$B(\Delta) = \frac{1}{R}\max\{(\Delta_u - \Delta)Su(1 + e_u(\Delta)) + B_u,\ (\Delta_d - \Delta)Sd(1 + e_d(\Delta)) + B_d\}.$$

Then the cost of the least cost super replicating portfolio is just the minimum of $h(\Delta)$ over all real Δ.

Now we show this function does have a minimum. Note if Δ is large positive

$$\begin{aligned} h(\Delta) &= \Delta S + \frac{\max\{(\Delta_u - \Delta)Su(1-\mu) + B_u,\ (\Delta_d - \Delta)Sd(1-\mu) + B_d\}}{R} \\ &= \Delta S + \frac{(\Delta_d - \Delta)Sd(1-\mu) + B_d}{R} \\ &= \frac{\Delta S}{R}[R - d(1-\mu)] + \frac{\Delta_d Sd(1-\mu) + B_d}{R}, \end{aligned}$$

which $\to \infty$ as $\Delta \to \infty$, and if Δ is large negative,

$$\begin{aligned} h(\Delta) &= \Delta S + \frac{(\Delta_u - \Delta)Su(1+\lambda) + B_u}{R} \\ &= \frac{\Delta S}{R}[R - u(1+\lambda)] + \frac{\Delta_u Su(1+\lambda) + B_u}{R}, \end{aligned}$$

which $\to \infty$ as $\Delta \to -\infty$. Since h is clearly continuous, this implies that its minimum is attained at some value Δ_0, which may or may not be unique.

Suppose first that

$$B(\Delta_0)R = (\Delta_u - \Delta_0)Su(1 + e_u(\Delta_0)) + B_u = (\Delta_d - \Delta_0)Sd(1 + e_d(\Delta_0)) + B_d$$

so that

$$\begin{aligned} \Delta_0 Su + B(\Delta_0)R &= \Delta_u Su + B_u + (\Delta_u - \Delta_0)Sue_u(\Delta_0), \\ \Delta_0 Sd + B(\Delta_0)R &= \Delta_d Sd + B_d + (\Delta_d - \Delta_0)Sde_d(\Delta_0). \end{aligned}$$

This means that $(\Delta_0, B(\Delta_0))$ are the holdings in a replicating portfolio so that in this case a replicating portfolio is a least cost super replicating portfolio.

The second possibility is that

$$B(\Delta_0)R = (\Delta_u - \Delta_0)Su(1 + e_u(\Delta_0)) + B_u > (\Delta_d - \Delta_0)Sd(1 + e_d(\Delta_0)) + B_d. \quad (5)$$

If $\Delta_0 = \Delta_u$, then this means

$$B(\Delta_0)R = B_u$$

and so $(\Delta_u, B_u/R)$ are the holdings in a least cost super replicating portfolio. Otherwise $\Delta_0 \neq \Delta_u$ and it follows from (5) that for Δ in an open interval containing Δ_0,

$$h(\Delta) = \Delta S + \frac{1}{R}[(\Delta_u - \Delta)Su(1 + e_u(\Delta_0)) + B_u]$$

which defines a straight line with slope

$$S\left(1 - \frac{u(1 + e_u(\Delta_0))}{R}\right).$$

This is certainly nonzero if $\Delta_0 < \Delta_u$ in which case Δ_0 could not be a minimum. It could only be zero if $\Delta_0 > \Delta_u$ and $R = u(1 + e_u(\Delta_0)) = u(1 - \mu)$. In this case we define

$$\sigma = \inf\{\Delta \geq \Delta_u : (\Delta_u - \Delta)Su(1 - \mu) + B_u > (\Delta_d - \Delta)Sd(1 + e_d(\Delta)) + B_d\}.$$

Clearly $\Delta_u \leq \sigma < \Delta_0$ and if $\sigma \leq \Delta \leq \Delta_0$,

$$h(\Delta) = h(\Delta_0).$$

This means that for $\sigma \leq \Delta \leq \Delta_0$, $(\Delta, B(\Delta))$ are the holdings in a least cost super replicating portfolio. So if $\sigma = \Delta_u$, then $(\Delta_u, B_u/R)$ are the holdings in a least cost super replicating portfolio. If $\sigma > \Delta_u$, then

$$B(\sigma)R = (\Delta_u - \sigma)Su(1 + e_u(\sigma)) + B_u = (\Delta_d - \sigma)Sd(1 + e_d(\sigma)) + B_d$$

and so $(\sigma, B(\sigma))$ are the holdings in a replicating portfolio. Hence when $\sigma > \Delta_u$, a replicating portfolio is a least cost super replicating portfolio.

The third possibility is that

$$B(\Delta_0)R = (\Delta_d - \Delta_0)Sd(1 + e_d(\Delta_0)) + B_d > (\Delta_u - \Delta_0)Su(1 + e_u(\Delta_0)) + B_u. \quad (6)$$

If $\Delta_0 = \Delta_d$, then this means

$$B(\Delta_0)R = B_d$$

and so $(\Delta_d, B_d/R)$ are the holdings in a least cost super replicating portfolio. Otherwise $\Delta_0 \neq \Delta_d$ and it follows from (6) that for Δ near Δ_0,

$$h(\Delta) = \Delta S + \frac{1}{R}[(\Delta_d - \Delta)Sd(1 + e_d(\Delta_0)) + B_d]$$

and hence for the same Δ

$$h'(\Delta) = S\left(1 - \frac{d(1 + e_d(\Delta_0))}{R}\right).$$

This is certainly nonzero if $\Delta_0 > \Delta_d$ in which case Δ_0 could not be a minimum. It could only be zero if $\Delta_0 < \Delta_d$ and $R = d(1 + e_d(\Delta_0)) = d(1 + \lambda)$. In this case we define

$$\sigma = \sup\{\Delta \leq \Delta_d : (\Delta_u - \Delta)Su(1 + e_u(\Delta)) + B_u < (\Delta_d - \Delta)Sd(1 + \lambda) + B_d\}.$$

Clearly $\Delta_0 < \sigma \leq \Delta_d$ and if $\Delta_0 \leq \Delta \leq \sigma$,

$$h(\Delta) = h(\Delta_0).$$

This means that for $\Delta_0 \leq \Delta \leq \sigma$, $(\Delta, B(\Delta))$ are the holdings in a least cost super replicating portfolio. If $\sigma = \Delta_d$, then $(\Delta_d, B_d/R)$ are the holdings in a least cost super replicating portfolio. If $\sigma < \Delta_d$, then

$$B(\sigma)R = (\Delta_u - \sigma)Su(1 + e_u(\sigma)) + B_u = (\Delta_d - \sigma)Sd(1 + e_d(\sigma)) + B_d$$

and so $(\sigma, B(\sigma))$ are the holdings in a replicating portfolio. Hence when $\sigma < \Delta_d$, a replicating portfolio is a least cost super replicating portfolio.

Thus the proof of the theorem is completed.

3. The Second Theorem.

In this section we determine exactly when the different cases given in Theorem 1 arise.

Theorem 2. *Consider a one-period model with contingent claim (Δ_u, B_u) in the up state and (Δ_d, B_d) in the down state and define*

$$a_u = \begin{cases} (\Delta_d - \Delta_u)Su(1 + \lambda) + B_d - B_u & \text{if } \Delta_u \geq \Delta_d \\ (\Delta_d - \Delta_u)Su(1 - \mu) + B_d - B_u & \text{if } \Delta_u < \Delta_d \end{cases}$$

and

$$a_d = \begin{cases} (\Delta_d - \Delta_u)Sd(1 - \mu) + B_d - B_u & \text{if } \Delta_u \geq \Delta_d \\ (\Delta_d - \Delta_u)Sd(1 + \lambda) + B_d - B_u & \text{if } \Delta_u < \Delta_d \end{cases}.$$

(a) *When the replicating portfolio is unique, it is a least cost super replicating portfolio unless*

$$R > u(1-\mu), \quad a_d < 0$$

when $(\Delta_u, B_u/R)$ *are the holdings in a least cost super replicating portfolio, or if*

$$R < d(1+\lambda), \quad a_u > 0$$

when $(\Delta_d, B_d/R)$ *are the holdings in a least cost super replicating portfolio.*

(b) *When the replicating portfolio is not unique, it is necessary that*

$$\Delta_u < \Delta_d, \quad d(1+\lambda) \geq u(1-\mu).$$

If

$$R \geq d(1+\lambda),$$

there exists at least one replicating portfolio with share holdings Δ *satisfying* $\Delta \leq \Delta_u$ *and all such replicating portfolios are least cost super replicating portfolios. If*

$$d(1+\lambda) \geq R \geq u(1-\mu),$$

there exists at least one replicating portfolio with share holdings Δ *satisfying* $\Delta_u \leq \Delta \leq \Delta_d$ *and all such replicating portfolios are least cost super replicating portfolios. If*

$$R \leq u(1-\mu),$$

there exists at least one replicating portfolio with share holdings Δ *satisfying* $\Delta \geq \Delta_d$ *and all such replicating portfolios are least cost super replicating portfolios.*

Proof. From the proof of Theorem 1, we see that in order to determine the least cost super replicating portfolio, we need to find the values of Δ which minimise

$$h(\Delta) = \Delta S + B(\Delta),$$

where

$$B(\Delta) = \frac{1}{R}\max\{(\Delta_u - \Delta)Su(1+e_u(\Delta)) + B_u,\ (\Delta_d - \Delta)Sd(1+e_d(\Delta)) + B_d\}.$$

Then the holdings in the least cost super replicating portfolio are $(\Delta, B(\Delta))$.

Note that

$$a_d \geq a_u \quad \text{when} \quad \Delta_u \geq \Delta_d,$$

$$a_u > a_d \quad \text{when} \quad \Delta_d > \Delta_u,\ u(1-\mu) > d(1+\lambda),$$

$$a_u < a_d \quad \text{when} \quad \Delta_d > \Delta_u,\ u(1-\mu) < d(1+\lambda)$$

and

$$a_u = a_d \quad \text{when} \quad \Delta_d > \Delta_u,\ u(1-\mu) = d(1+\lambda).$$

So there are the following cases:

$$\begin{array}{lllll}
\textit{Case 1} & : & \Delta_u \geq \Delta_d, & a_u > 0 & \\
\textit{Case 2} & : & \Delta_u \geq \Delta_d, & a_d \geq 0 \geq a_u & \\
\textit{Case 3} & : & \Delta_u \geq \Delta_d, & a_d < 0 & \\
\textit{Case 4} & : & \Delta_u < \Delta_d, & u(1-\mu) > d(1+\lambda), & a_d > 0 \\
\textit{Case 5} & : & \Delta_u < \Delta_d, & u(1-\mu) > d(1+\lambda), & a_u \geq 0 \geq a_d \\
\textit{Case 6} & : & \Delta_u < \Delta_d, & u(1-\mu) > d(1+\lambda), & a_u < 0 \\
\textit{Case 7} & : & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), & a_u > 0 \\
\textit{Case 8} & : & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), & a_d < 0 \\
\textit{Case 9} & : & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), & a_u = 0 < a_d \\
\textit{Case 10} & : & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), & a_d = 0 > a_u \\
\textit{Case 11} & : & \Delta_u < \Delta_d, & u(1-\mu) < d(1+\lambda), & a_d > 0 > a_u \\
\textit{Case 12} & : & \Delta_u < \Delta_d, & u(1-\mu) = d(1+\lambda), & a_u = a_d > 0 \\
\textit{Case 13} & : & \Delta_u < \Delta_d, & u(1-\mu) = d(1+\lambda), & a_u = a_d < 0 \\
\textit{Case 14} & : & \Delta_u < \Delta_d, & u(1-\mu) = d(1+\lambda), & a_u = a_d = 0.
\end{array}$$

It follows from Palmer (2001) that in Cases 1-8 and 12, 13 there is a unique replicating portfolio. In Cases 9, 10 there are two replicating portfolios, in Case 11 there are three and in Case 14 there are infinitely many.

We give detailed proofs for cases 1, 10, 11 and 14.

Case 1

$$\Delta_u \geq \Delta_d, \quad a_u > 0$$

To determine $B(\Delta)$, we consider

$$(\Delta_d-\Delta)Sd(1+e_d(\Delta))+B_d-(\Delta_u-\Delta)Su(1+e_u(\Delta))-B_u = S[u(1+e_u(\Delta))-d(1+e_d(\Delta))](\Delta-a(\Delta)),$$

where

$$a(\Delta) = \frac{\Delta_u Su(1+e_u(\Delta)) - \Delta_d Sd(1+e_d(\Delta)) + B_u - B_d}{Su(1+e_u(\Delta)) - Sd(1+e_d(\Delta))}.$$

If $\Delta \geq \Delta_u$,

$$\begin{aligned} \Delta - a(\Delta) & \geq \Delta_u - a(\Delta) \\ & = \Delta_u - \tfrac{\Delta_u Su(1-\mu) - \Delta_d Sd(1-\mu) + B_u - B_d}{Su(1-\mu) - Sd(1-\mu)} \\ & = \tfrac{(\Delta_d - \Delta_u) Sd(1-\mu) + B_d - B_u}{Su(1-\mu) - Sd(1-\mu)} \\ & = \tfrac{a_d}{Su(1-\mu) - Sd(1-\mu)} \\ & > 0. \end{aligned}$$

If $\Delta_u \geq \Delta \geq \Delta_d$,

$$\begin{aligned} \Delta - a(\Delta) & \geq \Delta_d - a(\Delta) \\ & = \Delta_d - \tfrac{\Delta_u Su(1+\lambda) - \Delta_d Sd(1-\mu) + B_u - B_d}{Su(1+\lambda) - Sd(1-\mu)} \\ & = \tfrac{(\Delta_d - \Delta_u) Su(1+\lambda) + B_d - B_u}{Su(1+\lambda) - Sd(1-\mu)} \\ & = \tfrac{a_u}{Su(1+\lambda) - Sd(1-\mu)} \\ & > 0. \end{aligned}$$

However, if $\Delta \leq \Delta_d$,

$$a(\Delta) = \alpha = \frac{\Delta_u Su(1+\lambda) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1+\lambda) - Sd(1+\lambda)},$$

where

$$\begin{aligned} \Delta_d - \alpha & = \Delta_d - \tfrac{\Delta_u Su(1+\lambda) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1+\lambda) - Sd(1+\lambda)} \\ & = \tfrac{(\Delta_d - \Delta_u) Su(1+\lambda) + B_d - B_u}{Su(1+\lambda) - Sd(1+\lambda)} \\ & = \tfrac{a_u}{Su(1+\lambda) - Sd(1+\lambda)} \\ & > 0. \end{aligned}$$

So

$$\Delta - a(\Delta) \begin{cases} > 0 & \text{if } \Delta > \alpha \\ = 0 & \text{if } \Delta = \alpha \\ < 0 & \text{if } \alpha > \Delta \end{cases},$$

where

$$\alpha < \Delta_d.$$

It follows that

$$B(\Delta) = \frac{1}{R}\begin{cases} (\Delta_d - \Delta)Sd(1-\mu) + B_d & \text{if } \Delta \geq \Delta_d \\ (\Delta_d - \Delta)Sd(1+\lambda) + B_d & \text{if } \Delta_d \geq \Delta \geq \alpha \\ (\Delta_u - \Delta)Su(1+\lambda) + B_u & \text{if } \alpha \geq \Delta \end{cases}.$$

Note that, since $\Delta = a(\Delta)$ if $\Delta = \alpha$, it follows that

$$B(\alpha)R = (\Delta_d - \alpha)Sd(1 + e_d(\alpha)) + B_d = (\Delta_u - \alpha)Su(1 + e_u(\alpha)) + B_u$$

and so

$$\begin{aligned} \alpha Su + B(\alpha)R &= \Delta_u Su + B_u + (\Delta_u - \alpha)Su(1 + e_u(\alpha)) \\ \alpha Sd + B(\alpha)R &= \Delta_d Sd + B_d + (\Delta_d - \alpha)Sd(1 + e_d(\alpha)). \end{aligned}$$

Thus $(\alpha, B(\alpha))$ is the replicating portfolio.

With a view to determining the least cost super replicating portfolio, we observe that the function $h(\Delta)$ is continuous piecewise linear with slope

$$\frac{S}{R}\begin{cases} R - d(1-\mu) > 0 & \text{if } \Delta \geq \Delta_d \\ R - d(1+\lambda) & \text{if } \Delta_d \geq \Delta \geq \alpha \\ R - u(1+\lambda) < 0 & \text{if } \alpha_d \geq \Delta \end{cases}.$$

So if

$$R > d(1+\lambda),$$

the minimum is unique and occurs at $\Delta = \alpha$; if

$$R = d(1+\lambda),$$

the minimum is not unique and occurs at Δ between α and Δ_d; if

$$R < d(1+\lambda),$$

the minimum is unique and occurs at $\Delta = \Delta_d$.

Case 10

$$\Delta_u < \Delta_d, \quad u(1-\mu) < d(1+\lambda), \quad a_d = 0 > a_u.$$

This time to determine $B(\Delta)$, we consider

$$(\Delta_d - \Delta)Sd(1 + e_d(\Delta)) + B_d - (\Delta_u - \Delta)Su(1 + e_u(\Delta)) - B_u = S\beta(\Delta)(\Delta - a(\Delta)),$$

where $a(\Delta)$ is as in Case 1 and

$$\beta(\Delta) = u(1 + e_u(\Delta)) - d(1 + e_d(\Delta)).$$

If $\Delta < \Delta_u$ (resp. $= \Delta_u$),

$$\beta(\Delta) = u(1 + \lambda) - d(1 + \lambda) > 0$$

and

$$\begin{aligned} \Delta - a(\Delta) & < (\text{resp. } =)\Delta_u - a(\Delta) \\ & = \Delta_u - \tfrac{\Delta_u Su(1+\lambda) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1+\lambda) - Sd(1+\lambda)} \\ & = \tfrac{(\Delta_d - \Delta_u)Sd(1+\lambda) + B_d - B_u}{Su(1+\lambda) - Sd(1+\lambda)} \\ & = \tfrac{a_d}{Su(1+\lambda) - Sd(1+\lambda)} \\ & = 0 \end{aligned}$$

so that

$$\Delta - a(\Delta) < 0$$

if $\Delta < \Delta_u$ and

$$\Delta - a(\Delta) = 0$$

if $\Delta = \Delta_u$.
If $\Delta_d \geq \Delta > \Delta_u$,

$$\beta(\Delta) = u(1 - \mu) - d(1 + \lambda) < 0$$

and

$$\begin{aligned} \Delta - a(\Delta) & > \Delta_u - a(\Delta) \\ & = \Delta_u - \tfrac{\Delta_u Su(1-\mu) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1-\mu) - Sd(1+\lambda)} \\ & = \tfrac{(\Delta_d - \Delta_u)Sd(1+\lambda) + B_d - B_u}{Su(1-\mu) - Sd(1+\lambda)} \\ & = \tfrac{a_d}{Su(1-\mu) - Sd(1+\lambda)} \\ & = 0. \end{aligned}$$

If $\Delta > \Delta_d$,

$$\beta(\Delta) = u(1 - \mu) - d(1 - \mu) > 0$$

and

$$a(\Delta) = \alpha = \frac{\Delta_u Su(1-\mu) - \Delta_d Sd(1-\mu) + B_u - B_d}{Su(1-\mu) - Sd(1-\mu)},$$

where

$$\begin{aligned} \Delta_d - \alpha &= \Delta_d - \frac{\Delta_u Su(1-\mu) - \Delta_d Sd(1-\mu) + B_u - B_d}{Su(1-\mu) - Sd(1-\mu)} \\ &= \frac{(\Delta_d - \Delta_u) Su(1-\mu) + B_d - B_u}{Su(1-\mu) - Sd(1-\mu)} \\ &= \frac{a_u}{Su(1-\mu) - Sd(1-\mu)} \\ &< 0. \end{aligned}$$

So

$$\beta(\alpha)(\Delta - a(\Delta)) \begin{cases} > 0 & \text{if } \Delta > \alpha \\ = 0 & \text{if } \Delta = \alpha \\ < 0 & \text{if } \alpha > \Delta > \Delta_u \\ = 0 & \text{if } \Delta = \Delta_u \\ < 0 & \text{if } \Delta_u > \Delta \end{cases},$$

where

$$\alpha > \Delta_d.$$

It follows that

$$B(\Delta) = \frac{1}{R} \begin{cases} (\Delta_d - \Delta) Sd(1-\mu) + B_d & \text{if } \Delta \geq \alpha \\ (\Delta_u - \Delta) Su(1-\mu) + B_u & \text{if } \alpha \geq \Delta \geq \Delta_u \\ (\Delta_u - \Delta) Su(1+\lambda) + B_u & \text{if } \Delta_u \geq \Delta \end{cases}.$$

Note that, since $\Delta = a(\Delta)$ if $\Delta = \alpha$ or Δ_u, it follows as in Case 1 that $(\alpha, B(\alpha))$ and $(\Delta_u, B(\Delta_u)) = (\Delta_u, B_u/R)$ are replicating portfolios.

The function $h(\Delta)$ is continuous piecewise linear with slope

$$\frac{S}{R} \begin{cases} R - d(1-\mu) > 0 & \text{if } \Delta \geq \alpha \\ R - u(1-\mu) & \text{if } \alpha \geq \Delta \geq \Delta_u \\ R - u(1+\lambda) < 0 & \text{if } \Delta_u \geq \Delta \end{cases}.$$

So if

$$R > u(1-\mu),$$

the minimum is unique and occurs at $\Delta = \Delta_u$; if

$$R = u(1-\mu),$$

the minimum is not unique and occurs at all Δ between α and Δ_u; if

$$R < u(1-\mu),$$

the minimum is unique and occurs at $\Delta = \alpha$.

Case 11

$$\Delta_u < \Delta_d, \quad u(1-\mu) < d(1+\lambda), \quad a_d > 0 > a_u.$$

As in Case 10, we consider

$$(\Delta_d - \Delta)Sd(1+e_d(\Delta)) + B_d - (\Delta_u - \Delta)Su(1+e_u(\Delta)) - B_u = S\beta(\Delta)(\Delta - a(\Delta)).$$

If $\Delta < \Delta_u$

$$\beta(\Delta) = u(1+\lambda) - d(1+\lambda) > 0$$

and

$$a(\Delta) = \alpha_1 = \frac{\Delta_u Su(1+\lambda) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1+\lambda) - Sd(1+\lambda)},$$

where

$$\begin{aligned} \Delta_u - \alpha_1 &= \Delta_u - \frac{\Delta_u Su(1+\lambda) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1+\lambda) - Sd(1+\lambda)} \\ &= \frac{(\Delta_d - \Delta_u)Sd(1+\lambda) + B_d - B_u}{Su(1+\lambda) - Sd(1+\lambda)} \\ &= \frac{a_d}{Su(1+\lambda) - Sd(1+\lambda)} \\ &> 0. \end{aligned}$$

If $\Delta_d \geq \Delta \geq \Delta_u$,

$$\beta(\Delta) = u(1-\mu) - d(1+\lambda) < 0$$

and

$$a(\Delta) = \alpha_2 = \frac{\Delta_u Su(1-\mu) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1-\mu) - Sd(1+\lambda)},$$

where

$$\begin{aligned} \Delta_u - \alpha_2 &= \Delta_u - \frac{\Delta_u Su(1-\mu) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1-\mu) - Sd(1+\lambda)} \\ &= \frac{(\Delta_d - \Delta_u)Sd(1+\lambda) + B_d - B_u}{Su(1-\mu) - Sd(1+\lambda)} \\ &= \frac{a_d}{Su(1-\mu) - Sd(1+\lambda)} \\ &< 0 \end{aligned}$$

and

$$\begin{aligned}\Delta_d - \alpha_2 &= \Delta_d - \tfrac{\Delta_u Su(1-\mu) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1-\mu) - Sd(1+\lambda)} \\ &= \tfrac{(\Delta_d - \Delta_u)Su(1-\mu) + B_d - B_u}{Su(1-\mu) - Sd(1+\lambda)} \\ &= \tfrac{a_u}{Su(1-\mu) - Sd(1+\lambda)} \\ &> 0.\end{aligned}$$

If $\Delta > \Delta_d$,

$$\beta(\Delta) = u(1-\mu) - d(1-\mu) > 0$$

and

$$a(\Delta) = \alpha_3 = \frac{\Delta_u Su(1-\mu) - \Delta_d Sd(1-\mu) + B_u - B_d}{Su(1-\mu) - Sd(1-\mu)},$$

where

$$\begin{aligned}\Delta_d - \alpha_3 &= \Delta_d - \tfrac{\Delta_u Su(1-\mu) - \Delta_d Sd(1-\mu) + B_u - B_d}{Su(1-\mu) - Sd(1-\mu)} \\ &= \tfrac{(\Delta_d - \Delta_u)Su(1-\mu) + B_d - B_u}{Su(1-\mu) - Sd(1-\mu)} \\ &= \tfrac{a_u}{Su(1-\mu) - Sd(1-\mu)} \\ &< 0.\end{aligned}$$

So

$$\beta(\Delta)(\Delta - a(\Delta)) \begin{cases} > 0 & \text{if } \Delta > \alpha_3 \\ = 0 & \text{if } \Delta = \alpha_3 \\ < 0 & \text{if } \alpha_3 > \Delta > \alpha_2 \\ = 0 & \text{if } \Delta = \alpha_2 \\ > 0 & \text{if } \alpha_2 > \Delta > \alpha_1 \\ = 0 & \text{if } \Delta = \alpha_1 \\ < 0 & \text{if } \alpha_1 > \Delta \end{cases},$$

where

$$\alpha_1 < \Delta_u < \alpha_2 < \Delta_d < \alpha_3.$$

It follows that

$$B(\Delta) = \frac{1}{R} \begin{cases} (\Delta_d - \Delta)Sd(1-\mu) + B_d & \text{if } \Delta \geq \alpha_3 \\ (\Delta_u - \Delta)Su(1-\mu) + B_u & \text{if } \alpha_3 \geq \Delta \geq \alpha_2 \\ (\Delta_d - \Delta)Sd(1+\lambda) + B_d & \text{if } \alpha_2 \geq \Delta \geq \alpha_1 \\ (\Delta_u - \Delta)Su(1+\lambda) + B_u & \text{if } \alpha_1 \geq \Delta \end{cases}.$$

Note that, since $\Delta = a(\Delta)$ if $\Delta = \alpha_i$, where $i = 1, 2, 3$, it follows as in the previous cases that $(\alpha_i, B(\alpha_i))$ for $i = 1, 2, 3$ are replicating portfolios.

The function $h(\Delta)$ is continuous piecewise linear with slope

$$\frac{S}{R}\begin{cases} R - d(1-\mu) > 0 & \text{if } \Delta \geq \alpha_3 \\ R - u(1-\mu) & \text{if } \alpha_2 \leq \Delta \leq \alpha_3 \\ R - d(1+\lambda) & \text{if } \alpha_1 \leq \Delta \leq \alpha_2 \\ R - u(1+\lambda) < 0 & \text{if } \alpha_1 \geq \Delta \end{cases}.$$

So if

$$R < u(1-\mu),$$

the minimum is unique and occurs at $\Delta = \alpha_3$; if

$$R = u(1-\mu),$$

the minimum is not unique and occurs at all Δ between α_2 and α_3; if

$$d(1+\lambda) > R > u(1-\mu),$$

the minimum is unique and occurs at $\Delta = \alpha_2$; if

$$R = d(1+\lambda),$$

the minimum is not unique and occurs at all Δ between α_1 and α_2; if

$$R > d(1+\lambda),$$

the minimum is unique and occurs at $\Delta = \alpha_1$.

Case 14

$$\Delta_u < \Delta_d, \quad u(1-\mu) = d(1+\lambda), \quad a_u = a_d = 0.$$

To determine $B(\Delta)$, we consider

$$(\Delta_d - \Delta)Sd(1 + e_d(\Delta)) + B_d - (\Delta_u - \Delta)Su(1 + e_u(\Delta)) - B_u,$$

which equals

$$a_u = a_d = 0$$

if

$$\Delta_u \leq \Delta \leq \Delta_d,$$

and equals

$$S\beta(\Delta)(\Delta - a(\Delta)),$$

if

$$\Delta < \Delta_u \quad \text{or} \quad \Delta > \Delta_d,$$

where $a(\Delta)$ and $\beta(\Delta)$ are as in the previous cases.
If $\Delta > \Delta_d$,

$$\beta(\Delta) = u(1-\mu) - d(1-\mu) > 0$$

and

$$\begin{aligned}
\Delta - a(\Delta) &> \Delta_d - a(\Delta) \\
&= \Delta_d - \tfrac{\Delta_u Su(1-\mu) - \Delta_d Sd(1-\mu) + B_u - B_d}{Su(1-\mu) - Sd(1-\mu)} \\
&= \tfrac{(\Delta_d - \Delta_u)Su(1-\mu) + B_d - B_u}{Su(1-\mu) - Sd(1-\mu)} \\
&= \tfrac{a_u}{Su(1-\mu) - Sd(1-\mu)} \\
&= 0.
\end{aligned}$$

If $\Delta < \Delta_u$

$$\beta(\Delta) = u(1+\lambda) - d(1+\lambda) > 0$$

and

$$\begin{aligned}
\Delta - a(\Delta) &< \Delta_u - a(\Delta) \\
&= \Delta_u - \tfrac{\Delta_u Su(1+\lambda) - \Delta_d Sd(1+\lambda) + B_u - B_d}{Su(1+\lambda) - Sd(1+\lambda)} \\
&= \tfrac{(\Delta_d - \Delta_u)Sd(1+\lambda) + B_d - B_u}{Su(1+\lambda) - Sd(1+\lambda)} \\
&= \tfrac{a_d}{Su(1+\lambda) - Sd(1-\mu)} \\
&= 0.
\end{aligned}$$

So

$$\beta(\alpha)(\Delta - a(\Delta)) \begin{cases} > 0 & \text{if } \Delta > \Delta_d \\ = 0 & \text{if } \Delta_u \le \Delta \le \Delta_d \\ < 0 & \text{if } \Delta_u > \Delta \end{cases}.$$

It follows that

$$B(\Delta) = \frac{1}{R} \begin{cases} (\Delta_d - \Delta)Sd(1-\mu) + B_d & \text{if } \Delta \ge \Delta_d \\ (\Delta_u - \Delta)Su(1-\mu) + B_u & \text{if } \Delta_u \le \Delta \le \Delta_d \\ (\Delta_u - \Delta)Su(1+\lambda) + B_u & \text{if } \Delta_u \ge \Delta \end{cases}.$$

Note that, since

$$(\Delta_d - \Delta)Sd(1 + e_d(\Delta)) + B_d = (\Delta_u - \Delta)Su(1 + e_u(\Delta)) + B_u$$

if $\Delta = \alpha$, where α is any number between Δ_d and Δ_u, it follows that $(\alpha, B(\alpha))$ is a replicating portfolio for all α in $[\Delta_d, \Delta_u]$.

The function $h(\Delta)$ is continuous piecewise linear with slope

$$\frac{S}{R}\begin{cases} R - d(1-\mu) > 0 & \text{if } \Delta \geq \Delta_d \\ R - u(1-\mu) = R - d(1+\lambda) & \text{if } \Delta_d \geq \Delta \geq \Delta_u \\ R - u(1+\lambda) < 0 & \text{if } \Delta_u \geq \Delta \end{cases}.$$

So if

$$R < u(1-\mu) = d(1+\lambda),$$

the minimum is unique and occurs at $\Delta = \Delta_d$; if

$$R = u(1-\mu) = d(1+\lambda),$$

the minimum is not unique and occurs at all Δ between Δ_u and Δ_d; if

$$R > u(1-\mu) = d(1+\lambda),$$

the minimum is unique and occurs at $\Delta = \Delta_u$.

Now we summarize the results in the remaining cases.

Cases 4, 7 , 9, 12

These are like case 1. In all the cases there is a replicating portfolio with share holdings α to the left of both Δ_u and Δ_d. This replicating portfolio is unique except in Case 9 when $(\Delta_d, B_d/R)$ is also a replicating portfolio. The conclusion is that if

$$R > d(1+\lambda),$$

the minimum is unique and occurs at $\Delta = \alpha$; if

$$R = d(1+\lambda),$$

the minimum is not unique and occurs at all Δ between α and Δ_d; if

$$R < d(1+\lambda),$$

the minimum is unique and occurs at $\Delta = \Delta_d$.

Cases 3, 6 , 8, 13

These are like case 10. In all the cases there is a replicating portfolio with share holdings α to the right of both Δ_u and Δ_d. This replicating portfolio is unique except in Case 10 when $(\Delta_u, B_u/R)$ is also a replicating portfolio. The conclusion is that if

$$R < u(1-\mu),$$

the minimum is unique and occurs at $\Delta = \alpha$; if

$$R = u(1-\mu),$$

the minimum is not unique and occurs at all Δ between α and Δ_u; if

$$R > u(1-\mu),$$

the minimum is unique and occurs at $\Delta = \Delta_u$.

Case 2

In this case the minimum is unique and occurs at $\Delta = \alpha$, where

$$\Delta_u \geq \alpha \geq \Delta_d$$

and $(\alpha, B(\alpha))$ is the unique replicating portfolio.

Case 5

In this case there is a unique replicating portfolio $(\alpha, B(\alpha))$, where

$$\Delta_u \leq \alpha \leq \Delta_d.$$

Note that $\alpha = \Delta_d$ if and only if $a_u = 0$ and $\alpha = \Delta_d$ if and only if $a_d = 0$. If

$$R > u(1-\mu),$$

the minimum is unique and occurs at $\Delta = \Delta_u$; if

$$R = u(1-\mu),$$

the minimum occurs at all Δ between α and Δ_u; if

$$u(1-\mu) > R > d(1+\lambda),$$

the minimum is unique and occurs at $\Delta = \alpha$; if

$$R = d(1 + \lambda),$$

the minimum occurs at all Δ between α and Δ_d; if

$$R < d(1 + \lambda),$$

the minimum is unique and occurs at $\Delta = \Delta_d$.

Looking through all the cases, we see that the theorem has been proved.

4. Further Developments.

The work in this paper is the first step in the project of determining the least cost super replicating portfolio of a contingent claim in a multi-period model. Work is in progress on the two-period case and the least cost super replicating portfolios for short calls and puts are in the process of being determined. However, at this juncture it is not clear how the results obtained so far can be extended to more periods.

References

Bensaid, B., Lesne, J.P., Pages, H., and Scheinkman, J. (1992): "Derivative Asset Pricing with Transaction Costs." *Mathematical Finance*, 2, 63-86.

Black, F. and Scholes, M. (1973): "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy,* 81, 637-659.

Boyle, P. and Vorst, T. (1992): "Option Replication in Discrete Time with Transaction Costs." *Journal of Finance*, 47, 271-293.

Davis, M.H.A., Panas, V.G., and Zariphopoulou, T. (1992): "European Option Pricing with Transaction Costs." *SIAM Journal of Control and Optimization*, 31, 470-498.

Grannan, R.E. and Swindle, G.H. (1996): "Minimizing Transaction Costs of Option Hedging Strategies." *Mathematical Finance*, 6, 341-364.

Hodges, S. and Neuberger, A. (1989): "Optimal Replication of Contingent Claims under Transaction Costs." *Review of Futures Markets*, 8, 222-239.

Leland, H.E. (1985): "Option Pricing and Replication with Transaction Costs." *Journal of Finance*, 40, 1283-1301.

Palmer, K.J. (2001): "Replicating and Super Replicating Portfolios in the Boyle-Vorst Discrete-Time Option Pricing Model with Transactions Costs." Research Paper 01-04, Department of Finance, University of Melbourne.

Rutkowski, M. (1998): "Optimality of Replication in the CRR Model with Transaction Costs." *Applicationes Mathematicae*, 25, 29-53.

Stettner, L. (1997): "Option Pricing in the CRR Model with Proportional Transaction Costs: A Cone Transformation Approach." *Applicationes Mathematicae*, 24, 475-514.

Toft, K.B. (1996): "On the Mean-Variance Tradeoff in Option Replication with Transactions Costs." *Journal of Financial and Quantitative Analysis*, 31, 233-263.