Dynamic Greeks

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Abstract: The sensitivity of a price function to changes in the parameters is given by its derivatives w.r.t. the parameters – the so-called Greeks. The Greeks are easily calculated when the price possesses a closed form expression. In any case we may compute the Greeks as solutions to differential equations derived from the differential equation of the price function by simply differentiating it w.r.t. the parameters. The existence of Greeks is the hard part. The idea extends to other dynamical entities. Some examples with numerical illustrations are given.

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1 Introduction

A. Terminology. In the finance literature the derivatives of a price function w.r.t. parameters in the contract or in the stochastic model are known as 'Greeks'. Presumably, they are called so simply because they are denoted by Greek letters. (One might ask: why aren't their numerical values called 'Arabics'.) An alternative term, conforming with usage in other quantitative disciplines, would be 'sensitivities', see Saltelli et al. (2000). The Greeks are useful because they tell us which assumptions in our model are the critical ones, and because they play a role in the context of hedging, see e.g. Björk (1998).

B. From static to dynamic Greeks. Existing literature about Greeks deal with cases where the price function is given by a closed form expression and the Greeks can be obtained simply by differentiating this expression w.r.t. the parameters. When a closed form expression does not exist, other methods must be invented. A recent paper by Kalashnikov and Norberg (2000) proposes the following method, designed by them for the context of life insurance mathematics, but applicable to any situation where the function we are interested in is the solution to some deterministic differential equation: upon differentiating through the differential equation w.r.t. some parameter, we obtain a differential equation for the corresponding Greek, which can be solved alongside the differential equation for the primitive function. Of course, the device works also in cases where explicit expressions are in hand, and is actually as efficient as the straightforward approach.

C. Outline of the study. In Section 2 we illustrate the technique in the framework of the simple Black-Scholes model, in which Greeks have been extensively studied. In Section 3 we apply it to a Markov chain Market where explicit formulas typically do not exist. In Section 4 we provide a numerical example based on the latter situation. Finally, in Section 5 we show, what seems to be taken for granted by everybody but – to our knowledge – has not been proved by anybody, that the Greeks actually exist in the Markov model.

2 The Black-Scholes market

A. The model. There are two basic assets, a bank account and a stock, whose prices at time t are denoted by B_t and S_t, respectively, and are governed by the dynamics

$$dB_t = B_t\, r\, dt\,,$$

and

$$dS_t = S_t\left(\alpha\, dt + \sigma dW_t\right)\,,$$

where r, α, σ are constants and W is a standard Brownian motion. The discounted values at time 0 of the prices are $\tilde{B}_t = B_t/B_t = 1$ (certainly a martingale) and $\tilde{S}_t = S_t/B_t$, with dynamics $d\tilde{B}_t = 0$ and

$$d\tilde{S}_t = \tilde{S}_t \sigma d\tilde{W}_t\,,$$

where $\tilde{W}_t = \frac{\alpha - r}{\sigma}t + W_t$. Under an equivalent martingale measure the process $\tilde{W}$ is a standard Brownian motion and $\tilde{S}$ is a martingale. Denote expectation under the martingale measure by $\tilde{\mathbb{E}}$.

B. Option prices. A European type of option is a contingent claim due at some fixed time T is a function of the form

$$h(S_T)\,. \tag{2.1}$$

We may also call it a T-claim. Its unique arbitrage-free price at time t is

$$f(S_t, t) = e^{-(T-t)\,r}\tilde{\mathbb{E}}\,[h(S_T)|\,S_t]\ .$$

The price function $f(s,t)$ admits the explicit expression

$$f(s,t) = e^{-(T-t)\,r} \int_{-\infty}^{\infty} h(se^{(T-t)\,r\,+\,\sigma\sqrt{T-t}w}) \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw\,. \tag{2.2}$$

It is also the solution to the differential equation

$$f_t(s,t) = f(s,t)\,r - f_s(s,t)\,rs - \frac{1}{2} f_{ss}(s,t)\,\sigma^2 \tag{2.3}$$

subject to

$$f(s,T) = h(s)\,, \qquad s > 0\,. \tag{2.4}$$

To save notation we have used subscripts to signify derivatives:

$$f_s = \frac{\partial}{\partial s} f\,,\ \ f_{ss} = \frac{\partial^2}{\partial s^2} f\,,\ \ \text{etc}\,.$$

C. Greeks. By tradition, the derivatives appearing in (2.3) are denoted by the following Greeks:

$$\Delta = f_s\,, \tag{2.5}$$
$$\Gamma = f_{ss}\,, \tag{2.6}$$
$$\Theta = f_t\,. \tag{2.7}$$

Two more Greeks are standard:

$$\rho = f_r\,, \tag{2.8}$$
$$\mathcal{V} = f_\sigma\,. \tag{2.9}$$

(The dependence of the price function f on the parameters r and σ is suppressed in the notation $f(s,t)$.)

Differentiating (2.3) and (2.4) w.r.t. r, we obtain the differential equation

$$\rho_t(s,t) = \rho(s,t)\,r + f(s,t) - \rho_s(s,t)\,rs - f_s(s,t)\,s - \frac{1}{2}\rho_{ss}(s,t)\,\sigma^2 \tag{2.10}$$

and the side condition

$$\rho(s,T) = 0 \qquad s > 0\,. \tag{2.11}$$

Similarly, differentiating w.r.t. σ, we obtain

$$\mathcal{V}_t(s,t) = \mathcal{V}(s,t)\, r - \mathcal{V}_s(s,t)\, rs - \frac{1}{2}\mathcal{V}_{ss}(s,t)\, \sigma^2 - f_{ss}(s,t)\, \sigma \tag{2.12}$$

and

$$\mathcal{V}(s,T) = 0 \qquad s > 0\,. \tag{2.13}$$

Now, to determine the price function and its Greeks, solve the differential equations (2.3), (2.10), and (2.12) subject to the side conditions. (The differential equations (2.5), (2.6), and (2.7) need not be explicitly stated and solved since the right hand side quantities are directly computed as part of the algorithm.)

D. Computation. No matter what the claim h looks like, the price admits an explicit expression (2.2), hence the same goes for the Greeks. Computation based on these formulas amount to numerical integration, which essentially is the same as solving some ordinary differential equation(s) numerically, see e.g. Los (2001). Alternatively, one may solve the partial differential equations (2.3), (2.10), and (2.12) numerically. Compared with the former method, the latter is just as easy to implement, is just a little bit slower (if you cannot spare a second), and is just a little bit less accurate (if you cannot spare a change).

3 The Markov chain market

A. The model. From Norberg (1999) we fetch the following, stripped here of details and ceremonial statements.

Let $\{Y_t\}_{t\geq 0}$ be a homogeneous continuous time Markov chain with finite state space $\mathcal{Y} = \{1, \ldots, n\}$ and infinitesimal matrix

$$\Lambda = (\lambda^{jk})\,.$$

For $j \neq k$, λ^{jk} is the intensity of transition from state j to state k, and $\lambda^{jj} = -\sum_{k;\, k\neq j} \lambda^{jk}$ is minus the total intensity of transition out of state j. To save notation we assume that all states communicate directly, that is, $\lambda^{jk} > 0$, $j \neq k$.

Introduce the indicator functions

$$I_t^j = 1[Y_t = j]\,,$$

and the counting processes

$$N_t^{jk} = |\{s;\, 0 < s \le t,\, Y_{s-} = j\,,\, Y_s = k\}|\,.$$

Letting Y_t represent the state of the economy at time t, we introduce a market with $n+1$ basic traded assets:

Asset No. 0 is a "locally risk-free" *bank account* with price process B_t given by

$$dB_t = B_t \sum_j r^j I_t^j \, dt\,,$$

where the r^j are constants. Thus, the short rate of interest is

$$r_t = \sum_j r^j I_t^j\,,$$

and the r^j are the state-wise interest rates.

The remaining n assets are risky *stocks*. The price process S_t^i of stock No. i has dynamics

$$dS_t^i = S_{t-}^i \left(\sum_j \alpha^{ij} I_t^j \, dt + \sum_j \sum_{k;\, k \ne j} \gamma^{ijk} dN_t^{jk} \right)\,, \tag{3.14}$$

$i = 1, \ldots, n$, where the α^{ij} and γ^{ijk} are constants.

The discounted stock prices, $\tilde{S}_t^i = \tilde{S}_t^i / B_t$, have dynamics

$$d\tilde{S}_t^i = \tilde{S}_{t-}^i \left(\sum_j (\alpha^{ij} - r^j) I_t^j \, dt + \sum_j \sum_{k;\, k \ne j} \gamma^{ijk} dN_t^{jk} \right)\,. \tag{3.15}$$

Assume there exists an infinitesimal matrix

$$\tilde{\Lambda} = (\tilde{\lambda}^{jk})\,,$$

which is equivalent to Λ in the sense that $\tilde{\lambda}^{jk} = 0$ if and only if $\lambda^{jk} = 0$, and such that

$$\sum_{k;\, k \ne j} \gamma^{ijk} \tilde{\lambda}^{jk} = r^j - \alpha^{ij}\,, \tag{3.16}$$

$i = 1, \ldots, n$, $j = 1, \ldots, n$. Then (3.15) can be written as

$$d\tilde{S}_t^i = \tilde{S}_{t-}^i \left(\sum_j \sum_{k;\, k \ne j} \gamma^{ijk} d\tilde{M}_t^{jk} \right)\,,$$

where the processes $\tilde{M}^{jk}$, $j \neq k$, are given by

$$d\tilde{M}^{jk}_t = dN^{jk}_t - I^j_t \tilde{\lambda}^{jk}\, dt\,.$$

Under the equivalent measure with infinitesimal matrix $\tilde{\Lambda}$ the $\tilde{M}^{jk}$ are the compensated counting processes, which are zero mean, square integrable, mutually orthogonal martingales. Thus, the discounted price processes are martingales under the equivalent measure, and so the market admits no arbitrage.

Assume, moreover, that the coefficient matrix $\Gamma = (\gamma^{jk})$ (with diagonal elements defined as 0) has full rank. Then the market is also complete.

B. Option prices. Consider a contract which pays $h^{Y_T}(S^\ell_T)$ at time $T > 0$; the payment depends on the state of the economy and the price of stock No. ℓ at the term T. Due to its special role, we elevate the state variable to a top-script; basically, we are dealing with a vector-valued function $\mathbf{h}(s) = (h^1(s), \ldots, h^n(s))'$.

The unique arbitrage-free price of the claim at time $t < T$ is

$$f^{Y_t}(S^\ell_t, t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r_s\, ds} h^{Y_T}(S^\ell_T) \,\middle|\, Y_t,\, S^\ell_t \right]. \qquad (3.17)$$

Explicit formulas exist for zero coupon bonds and for claims of the simple form $h(Y_t)$, e.g. caplets and other interest derivatives, see Norberg (1999). (They involve the exponential function of a matrix, which is an infinite sum, and is in this sense just as 'explicit' as the exponential function of a real.) For stock derivatives one usually has to resort to simulation or numerical solution of differential equations. We take the latter approach:

The discounted price $e^{-\int_0^t r_s\, ds} f^{Y_t}(S^\ell_t, t)$ is a martingale under the equivalent measure. Operating on it with Itŏ's formula and identifying the drift term (which must be null), we arrive at the following system of first order partial differential equations for $\mathbf{f}(s,t) = (f^1(s,t), \ldots, f^n(s,t))'$:

$$\begin{aligned} f^j_t(s,t) &= r^j f^j(s,t) - f^j_s(s,t)\, s\, \alpha^{\ell j} \\ &\quad - \sum_{k;\; k \neq j} \left(f^k(s(1+\gamma^{\ell jk}), t) - f^j(s,t) \right) \tilde{\lambda}^{jk}\,. \end{aligned} \qquad (3.18)$$

These are to be solved subject to the conditions

$$f^j(s,T) = h^j(s)\,, \qquad (3.19)$$

$j = 1, \ldots, n$.

For claims of the simple form $h(Y_T)$, (3.18) reduces to the ordinary first order differential equations

$$\frac{d}{dt} f^j(t) = r^j f^j(t) - \sum_{k;\, k \neq j} \left(f^k(t) - f^j(t) \right) \tilde{\lambda}^{jk} , \tag{3.20}$$

with conditions

$$f^j(T) = h^j , \tag{3.21}$$

$j = 1, \ldots, n$.

C. Greeks. Consider the stock option price **f** discussed in Paragraph B above, which (except is some very simple special cases) has to be determined as the solution to the differential equation (3.18) with the side condition (3.19). Greeks are straightforwardly obtained as solutions to the differential equations and side conditions obtained upon differentiating (3.18) and (3.19). For an example, take the sensitivity of the price function **f** with respect to the drift parameter $\alpha^{\ell h}$ (the drift coefficient of the stock in state h). No traditional name exists for this sensitivity, of course, so let us denote it by the Greek letter ν (ν_h might be better). We find

$$\begin{aligned} \nu^j_t(s,t) &= r^j \nu^j(s,t) - \nu^j_s(s,t)\, s\, \alpha^{\ell j} - f^j_s(s,t)\, s \\ &\quad - \sum_{k;\, k \neq j} \left(\nu^k(s(1+\gamma^{\ell jk}), t) - \nu^j(s,t) \right) \tilde{\lambda}^{jk} \\ &\quad - \sum_{k;\, k \neq j} \left(\nu^k(s(1+\gamma^{\ell jk}), t) - \nu^j(s,t) \right) \frac{\partial}{\partial \alpha^{\ell j}} \tilde{\lambda}^{jk} , \end{aligned} \tag{3.22}$$

$$\nu^j(s,T) = 0 , \tag{3.23}$$

$j = 1, \ldots, n$. The derivatives of the $\tilde{\lambda}^{jk}$ w.r.t. $\alpha^{\ell j}$ are obtained upon differentiating (3.16).

D. Computation. Numerical solution of the differential equations (3.18) and (3.22) goes (simultaneously) by the Lax-Wendroff difference scheme, modified to account of the non-standard feature that the differential equation involves f-values at different values of the s-argument.

The ordinary differential equations (3.20) are easily and accurately solved by standard methods like Runge-Kutta.

4 An example

As an illustration we take the simple example of a European call option with maturity T and strike K, $h(S_T) = (S_T - K)_+$, in a simple Poisson analogue to the Black-Scholes scenario of Section 2. The bank account is the same as before, but the price process of the stock is now

$$S_t = \exp\left(\alpha t + \sigma N_t\right) ,$$

where α and σ are constants and N is a counting process with strictly positive intensity. By the general Itŏ's formula the discounted stock price has dynamics

$$d\tilde{S}_t = \tilde{S}_{t-}\left((\alpha - r)\, dt + \gamma dN_t\right) ,$$

where $\gamma = e^{\sigma} - 1$. The equivalent martingale measure is seen to be the one under which N is a homogeneous Poisson process with intensity $\tilde{\lambda} = (r-\alpha)/\gamma$. (The physical intensity of N does not matter, of course, since it is not a path property of the process.) This model can be accommodated in the Markov chain framework above upon specifying two states with intensities of transition $\tilde{\lambda}^{12} = \tilde{\lambda}^{21} = \tilde{\lambda}$, and one stock with $\alpha^{11} = \alpha^{21} = \alpha$ and $\gamma^{112} = \gamma^{121} = \gamma$. The price function is now independent of the state, and we call it just $f(s,t)$.

The price of the call option at time t is

$$f(s,t) = e^{-r(T-t)} \sum_{n=0}^{\infty} \left(s e^{\alpha(T-t)+\sigma n} - K\right)_+ \frac{\left(\tilde{\lambda}(T-t)\right)^n}{n!} e^{-\tilde{\lambda}(T-t)} . \quad (4.24)$$

The differential equation (3.18) and the side condition (3.19) reduce to

$$f_t(s,t) = r\, f(s,t) - f_s(s,t)\, s\, \alpha - \left(f(s(1+\gamma),t) - f(s,t)\right) \tilde{\lambda} , \quad (4.25)$$

$$f(s,T) = (s-K)_+ . \quad (4.26)$$

The sensitivity $\nu(s,t) = \frac{\partial}{\partial\alpha} f(s,t)$ is the solution to

$$\begin{aligned} \nu_t(s,t) &= r\, \nu(s,t) - nu_s(s,t)\, s\, \alpha - f_s(s,t)\, s - \left(\nu(s(1+\gamma),t) - \nu(s,t)\right) \tilde{\lambda} \\ &\quad + \left(f(s(1+\gamma),t) - f(s,t)\right) \alpha/\gamma , \end{aligned} \quad (4.27)$$

$$\nu(s,T) = 0 . \quad (4.28)$$

Actually, inspecting (4.24) one realizes that the derivatives involved in (4.27) do not exist on those curves in the positive quadrant of the (s,t)-plane where

$se^{\alpha(T-t)+\sigma n} = K$ for some integer n because at such points an additional term is being added to the sum (4.24). None the less the difference scheme works well since it essentially only requires continuity and piece-wise differentiability.

The following results were obtained for $r = \ln(1.05)$, $\alpha = 0.045$, $\gamma = 0.02$, $T = 1$, $K = 1.05$, and $S_0 = 1$: The sensitivities of the price at time 0 of the European call option are 0.85 w.r.t. r, -0.70 w.r.t. α, and 0.040 w.r.t. γ. (One should think about the following consequence: Increasing α and hence the performance of the stock, makes the option worth less. Increasing γ and hence the performance of the stock, makes the option worth more.)

5 Greeks exist

We will focus here on differentiability w.r.t. parameters in the probability model, which is the hard part.

Consider a family of probability measures $\{\mathbb{P}_\theta;\, \theta \in \Theta\}$ indexed by a parameter θ in some open finite-dimensional Euclidean set. To fix ideas, let us have in mind continuous time Markov chain with finite state space and intensities that are parametric functions, $\mu_\theta^{jk}(t)$. Being mainly interested in the first order derivative in one direction at a time, we can as well assume that θ is real-valued and assumes its values in an open interval. Denote the infinitesimal matrix by $\mathbf{M}_\theta(t)$, where t is time, and denote the corresponding matrix of transition probabilities over the time interval from t to u by $\mathbf{P}_\theta(t,u)$.

Using classical techniques, Kalashnikov and Norberg (2000) proved that, if $\mathbf{M}_\theta(t)$ is sufficiently smooth, then $\mathbf{P}_\theta(t,u)$ is differentable w.r.t. θ and

$$\frac{\partial}{\partial\theta}\mathbf{P}_\theta(t,u) = \int_t^u \mathbf{P}_\theta(t,\tau)\,\frac{\partial}{\partial\theta}\mathbf{M}_\theta(\tau)\,\mathbf{P}_\theta(\tau,u)\,d\tau\,. \tag{5.29}$$

We will sketch the proof of result that is more general, at the expense of introducing the additional assumption that the probability measures are mutually absolutely continuous: for fixed j, k, and t either $\mu_\theta^{jk}(t) > 0$ for all θ or $\mu_\theta^{jk}(t) = 0$ for all θ.

By Girsanov's theorem for counting processes (see e.g. Andersen et al. 1993),

$$\mathbf{E}_{\theta+\eta}[X] = \mathbf{E}_\theta[X\,L_{\theta+\eta,\theta}(T)]$$

where $L_{\theta+\eta,\theta}(T)$ is the likelihood of $\mathbb{P}_{\theta+\eta}$ with respect to $\mathbb{P}_\theta$. It is the value at time T of the likelihood process, which is a martingale under $\mathbb{P}_\theta$, given by $L_{\theta+\eta,\theta}(0) = 1$ and the dynamics

$$dL_{\theta+\eta,\theta}(t) = L_{\theta+\eta,\theta}(t-)\frac{\mu^{jk}_{\theta+\eta}(t) - \mu^{jk}_{\theta}(t)}{\mu^{jk}_{\theta}(t)}\, dM^{jk}_{\theta}(t)\,, \tag{5.30}$$

where the M^{jk}_{θ} are the compensated counting processes given by

$$dM^{jk}_{\theta}(t) = dN^{jk}(t) - I^{j}(t)\mu^{jk}_{\theta}(t)\, dt\,.$$

They are square integrable and mutually orthogonal martingales under $\mathbb{P}_{\theta}$.

Let X be an integrable $\mathcal{F}_T$-measurable random variable. Under $\mathbb{P}_{\theta}$ it has the martingale representation

$$X = \mathbf{E}_{\theta}[X] + \int_0^T \sum_{j\neq k} \xi^{jk}_{\theta}(\tau) dM^{jk}_{\theta}(\tau)\,. \tag{5.31}$$

We have

$$\begin{aligned}
&\frac{1}{\eta}\left(\mathbf{E}_{\theta+\eta}[X \mid \mathcal{F}_t] - \mathbf{E}_{\theta}[X \mid \mathcal{F}_t]\right)\\
&= \frac{1}{\eta}\left(\frac{\mathbf{E}_{\theta}[X L_{\theta+\eta,\theta}(T) \mid \mathcal{F}_t]}{L_{\theta+\eta,\theta}(t)} - \mathbf{E}_{\theta}[X \mid \mathcal{F}_t]\right)\\
&= \frac{1}{\eta\, L_{\theta+\eta,\theta}(t)}\mathbf{Cov}_{\theta}[X, L_{\theta+\eta,\theta}(T) \mid \mathcal{F}_t]\,.
\end{aligned}$$

Using the martingale representations (5.31) and (5.30), we continue:

$$\begin{aligned}
&= \frac{1}{\eta\, L_{\theta+\eta,\theta}(t)}\mathbf{E}_{\theta}\left[\int_t^T \sum_{j\neq k}\xi^{jk}_{\theta}(\tau) L_{\theta+\eta,\theta}(\tau-)\frac{\mu^{jk}_{\theta+\eta}(\tau) - \mu^{jk}_{\theta}(\tau)}{\mu^{jk}_{\theta}(\tau)} d\langle M^{jk}_{\theta}\rangle(\tau)\,\middle|\, \mathcal{F}_t\right]\\
&= \int_t^T \sum_{j\neq k}\mathbf{E}_{\theta}\left[\xi^{jk}_{\theta}(\tau)\frac{L_{\theta+\eta,\theta}(\tau)}{L_{\theta+\eta,\theta}(t)}\frac{\mu^{jk}_{\theta+\eta}(\tau) - \mu^{jk}_{\theta}(\tau)}{\eta} I^{j}(\tau)\,\middle|\, \mathcal{F}_t\right] d\tau\,.
\end{aligned}$$

For $\theta + \eta$ in an ϵ-neighbouthood of θ the integrand here is bounded by

$$\sum_{j\neq k}\mathbf{E}_{\theta}\left[\xi^{jk}_{\theta}(\tau)\frac{L_{\theta+\eta,\theta}(\tau)}{L_{\theta+\eta,\theta}(t)} \sup_{\theta^{*}\in(\theta-\epsilon,\theta+\epsilon)}\frac{\partial}{\partial\theta}\mu^{jk}_{\theta^{*}}(\tau)\,\middle|\, \mathcal{F}_t\right]\,.$$

One may now invent various conditions of boundedness and/or integrability to justify taking the limit under the integral sign as η goes to 0, and conclude that $\mathbf{E}_\theta[X]$ is differentiable w.r.t. θ and

$$\frac{\partial}{\partial\theta}\mathbf{E}_\theta[X] = \mathbf{E}_\theta\left[X\int_0^T \sum_{g\neq h}\frac{\partial}{\partial\theta}\ln\mu_{gh}(\tau,\theta)dM_{gh}(\tau,\theta)\right]. \tag{5.32}$$

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