Hazard rate for credit risk and hedging defaultable contingent claims

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Abstract

We provide a concise exposition of theoretical results that appear in modelling default time as a random time, and we establish representation theorems. We focus on the following issues: the completion of the defaultable market, the evaluation and the hedging of defaultable contingent claims.

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1 Introduction

The aim of main papers on default risk is to compare prices of default-free contingent claims and defaultable ones, and their commun starting point is the knowledge of the default-free assets' dynamics and of the default process. Following this methodology, we investigate the links between the default time and the default-free information $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$. The filtration $\mathbf{F}$ is generally included in the filtration of default-free assets' prices and τ is a random time, that is a non-negative random variable. In the so-called structural approach, τ is a stopping time in the filtration $\mathbf{F}^S$ generated by default-free assets' prices and it is assumed that the agents have all the information contained on prices, i.e., $\mathbf{F} = \mathbf{F}^S$, whereas in the reduced-form approach, the default arrives "by surprise", for example as in Cox process modelling (See Cooper and Martin [?] or the more recent paper of Rogers [?] for a survey on these two approaches.) An intermediary case is when τ is an $\mathbf{F}^S$-stopping time and $\mathbf{F} \subset \mathbf{F}^S$, for example when the filtration $\mathbf{F}$ is the trivial one or when $\mathbf{F}$ is the filtration generated by the information of prices observed at discrete times [?] . We give a representation theorem in a general setting and we discuss the choice of tradeable assets in the defaultable market. We investigate in particular the so-called Cox process approach, which enjoys the characteristic invariance property that $\mathbf{F}$-martingales are immerged in $\mathbf{G}$-martingales. When the default-free market is complete we make precise the hedging of defaultable contingent claim using defaultable zero-coupons and default-free assets. In particular, we prove that the hedging strategy of a defaultable terminal payoff consists of a self-financing strategy with t-time value V_t which invests exactly V_t in the defaultable zero-coupon.

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2 The model

We study a model of financial market where a riskless asset is traded, as well as financial assets. We suppose that some probability space $(\Omega, \mathcal{G}, P)$ is given. We limit our study, mainly for simplicity of notation, to the case where there are only one financial asset (real-valued), whose price at time t is denoted by the random variable S_t and a riskless asset with deterministic interest rate $(r(s), s \geq 0)$. We denote by $R_t = \exp\left(-\int_0^t r(s)ds\right)$ the discounted factor and by $S_t^0 = \exp\left(\int_0^t r(s)ds\right)$ the savings account. In particular, the zero-coupon of maturity T has price $B(t,T) = \exp\left(-\int_t^T r(s)ds\right)$. We shall refer to this asset as the *default-free zero-coupon* and to the market where the default-free zero coupon and the risky asset S are traded till to maturity T as the *default-free market.* The filtration generated by the price of the risky asset is denoted $\mathbf{F}^S = (\mathcal{F}_t^S = \sigma(S_s, s \leq t); t \geq 0)$.

A default occurs at a random time (i.e., a non-negative random variable on the space $(\Omega, \mathcal{G}, P)$) denoted by τ. In the defaultable world, the payment of contingent claims depends whether or not the default has appeared before the maturity. In particular, we shall study defaultable zero-coupon with maturity T (in short DZC) which pays 1 unit at maturity if and only if the default has not appeared before T. More generally, we investigate the case where a promised payof X is paid at maturity if and only if the defaut has not appeared before maturity. In case where the default occurs before maturity, a rebate can be paid at hit (i.e., at default time) or at maturity. The rebate can be a function of the default time τ, or of the value of the asset at time τ.

2.1 Three important filtrations

We assume, as in [**?**, **?**, **?**] that the t-time information available in the default-free market is a σ-algebra $\mathcal{F}_t$. We do not assume that the filtration $\mathbf{F} = (\mathcal{F}_t; t \geq 0)$ modelling this information is the filtration $\mathbf{F}^S$ generated by the risky asset S. The filtration $\mathbf{F}$ can be smaller or larger than $\mathbf{F}^S$.
We denote by $\mathbf{D}$ the filtration $\mathbf{D} = (\mathcal{D}_t; t \geq 0)$ with $\mathcal{D}_t = \sigma(D_s, s \leq t)$ where D is the default process defined as $D_t = 1\!\!1_{\{\tau \leq t\}}$. At time t, the information on the price and on default time is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$: at any time the agent knows whether or not the default has appeared. Hence, the default time τ is a $\mathbf{G}$-stopping time where $\mathbf{G} = (\mathcal{G}_t,\, t \geq 0)$. In fact, $\mathbf{G}$ is the smallest filtration which contains $\mathbf{F}$, satisfying the usual hypotheses, such that τ is a $\mathbf{G}$-stopping time.

2.2 Hazard process

In this section, we work under a reference probability P. Latter on, this probability will be either the historical probability, or a risk neutral one.
Let F be the right-continuous version of the submartingale $F_t = P(\tau \leq t|\mathcal{F}_t)$ and G the conditional survival probability $G_t = 1 - F_t$. We introduce the $I\!\!R^+ \cup \{+\infty\}$-valued *hazard process*[1] $\Gamma_t = -\ln(G_t)$. Setting as usual $e^{-\infty} = 0$, we have $G_t = e^{-\Gamma_t}$. We assume for simplicity that $F_0 = 0$ so that $\Gamma_0 = 0$. For typographic reasons, we shall often use F, G and Γ in the same formula. The $\mathbf{F}$-stopping time

[1] If needed, we shall say the P-$\mathbf{F}$ hazard process

$\Upsilon = \inf\{t : F_t = 1\}$, introduced in Andreasen [?] in order to model the explosion time of intensity process, satisfies $\tau \leq \Upsilon$, so that the hazard process is finite on the set $\{t < \tau\}$.

We recall a key lemma established in Dellacherie [?, ?] and its corollary.

Lemma 1 *Let $X \in \mathcal{F}_T$ be integrable. Then,*

$$E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}\, e^{\Gamma_t} E(X G_T|\mathcal{F}_t)\,. \tag{1}$$

Corollary 1 *Let h be a* **F***-predictable bounded process. Then,*
a)

$$E(h_\tau|\mathcal{G}_t) = 1\!\!1_{\{\tau\leq t\}} h_\tau + 1\!\!1_{\{t<\tau\}}\, e^{\Gamma_t} E\Big(\int_t^\infty h_u dF_u\,|\mathcal{F}_t\Big)\,. \tag{2}$$

b) In particular, if F is increasing and continuous,

$$E(h_\tau|\mathcal{G}_t) = 1\!\!1_{\{\tau\leq t\}} h_\tau + 1\!\!1_{\{t<\tau\}}\, E\Big(\int_t^\infty h_u \exp\big(\Gamma_t - \Gamma_u\big)\, d\Gamma_u\,|\mathcal{F}_t\Big)\,. \tag{3}$$

The proof of this lemma and its corollary is based on the important remark that any $\mathcal{G}_t$-measurable random variable is equal, on the set $\{t < \tau\}$, to an $\mathcal{F}_t$-measurable random variable.
It is also useful to note that for any **G**-predictable process h, there exists an **F**-predictable process h^* such that both processes are equal on the set $\{t \leq \tau\}$, i.e., $h_t 1\!\!1_{\{t\leq\tau\}} = h^*_t 1\!\!1_{\{t\leq\tau\}}$. Moreover, under the hypothesis $\forall t, F_t < 1$ (i.e., $\Upsilon = \infty$), the process $(h^*_t, t \geq 0)$ is unique (See [?], page 186). Otherwise, this process is not unique on the set $\{t \geq \Upsilon\}$.

Remark 1 In the very particular case when τ is an **F**-stopping time, formulae (**??**) and (**??**) are obvious. Indeed, in that case, $F_t = 1\!\!1_{\{\tau\leq t\}}$, the two filtrations **F** and **G** are equal and $\int_0^\infty h_u dF_u = h_\tau$. Hence, $G_t = 1\!\!1_{\{t<\tau\}}$, and the straightforward equalities

$$\begin{aligned} E(X 1\!\!1_{T<\tau}|\mathcal{F}_t) &= 1\!\!1_{\{t<\tau\}} E(X 1\!\!1_{\{T<\tau\}}|\mathcal{F}_t) \\ E(h_\tau|\mathcal{F}_t) &= h_\tau 1\!\!1_{\{\tau\leq t\}} + 1\!\!1_{\{t<\tau\}} E(\int_t^\infty h_u dF_u|\mathcal{F}_t) \end{aligned}$$

are exactly (**??**) and (**??**).

2.3 A basic martingale

As an application of lemma **??**, it is easy to check that the process $(L_t, t \geq 0)$ where

$$L_t = 1\!\!1_{t<\tau} e^{\Gamma_t} = (1 - D_t)\, e^{\Gamma_t}$$

is a **G**-martingale. This non-negative martingale is obviously discontinuous.
From lemma (**??**), for any $X \in \mathcal{F}_T$

$$E(1\!\!1_{T<\tau} R_T X|\mathcal{G}_t) = L_t E(R_T G_T X|\mathcal{F}_t)\,. \tag{4}$$

Remark 2 If we set $dQ|_{\mathcal{G}_t} = L_t dP|_{\mathcal{G}_t}$ the probability Q is absolutely continuous with respect to P and

$$\forall X \in \mathcal{F}_T,\ E_Q(X|\mathcal{G}_t) = 1\!\!1_{t<\tau} E_P(X|\mathcal{F}_t)\,.$$

Indeed, setting $L_T^t = 1\!\!1_{t<\tau} L_T / L_t$

$$E_Q(X|\mathcal{G}_t) = E_P(L_T^t X|\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Gamma_t} E_P(e^{\Gamma_T - \Gamma_t} X e^{-\Gamma_T}|\mathcal{G}_t)\,.$$

However, due to the fact that we change as well the probability and the filtration, this change of measure seems to be not very useful.

3 Enlargement of filtration

One major question is to describe the dynamics of the risky asset S in the filtration $\mathbf{G}$. As mentioned in Hull and White [?] " When we move from the vulnerable world to a default free world, the stochastic processes followed by the underlying state variables may change." We are studying here the reverse case, i.e., we move from the default-free world to the vulnerable one. We establish in this section a general representation theorem for some $\mathbf{G}$ martingales. Then, we discuss the financial implication of that theorem.

3.1 Decomposition of the F-martingales as G-semi-martingales

We recall some facts on enlargement of filtration. The submartingale F admits a unique Doob-Meyer decomposition as $F_t = Z_t + A_t$, where Z is an $\mathbf{F}$-martingale and A an $\mathbf{F}$-predictable increasing process. Moreover, the process $M_t = D_t - \Lambda_{t\wedge\tau}$ where $d\Lambda_t = dA_t/G_{t-}$ is a $\mathbf{G}$-martingale. (See e.g. [?], [?] and [?] for proofs and comments.)

Moreover, we require that (C) holds, where

(C) One of the following conditions is satisfied

(i) Any $\mathbf{F}$-martingale is continuous

(ii) For any $\mathbf{F}$-stopping time θ, $P(\tau = \theta) = 0$.

Under this condition, an $\mathbf{F}$-martingale has no common jump with the hazard process.

We denote by $[X, Y]$ the quadratic covariation of two semi-martingales X, Y and by $[X]$ the quadratic variation of the semi-martingale X.

If (C) holds, it is well known (see Dellacherie et al. [?], page 188 or Yor [?], page 41) that for any $\mathbf{F}$-martingale m, the process

$$\widehat{m}_{t\wedge\tau} = m_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{\Gamma_s} d[m, Z]_s$$

is a stopped $\mathbf{G}$-martingale. We shall refer to $\widehat{m}$ as the $\mathbf{G}$-martingale part of m (which is an $\mathbf{F}$-martingale and a $\mathbf{G}$-semi martingale.)

Remark 3 When m is continuous, the martingale property of $\widehat{m}$ can be checked using corollary **??**.

3.2 Representation theorem

Theorem 1 *Suppose that* (C) *holds and let $F = Z + A$ be the Doob-Meyer decomposition of F. Let h be an $\mathbf{F}$-predictable process such that $\int_0^\infty h_u dF_u$ is an integrable r.v., and let $H_t = E(h_\tau|\,\mathcal{G}_t)$. Then, the process H admits a decomposition in martingales as follows*

$$H_t = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_{s-}} (d\widehat{m}_s^h - (h_s - J_{s-}^h) d\widehat{Z}_s) + \int_{]0,t\wedge\tau]} e^{\Delta\Gamma_s} (h_s - J_{s-}^h)\, dM_s\,. \tag{5}$$

Here m^h is the **F***-martingale*

$$m_t^h = E\Big(\int_0^\infty h_u dF_u |\, \mathcal{F}_t\Big) = E\Big(\int_0^\infty h_u dA_u |\, \mathcal{F}_t\Big)\,,$$

$\widehat{m}^h$ and $\widehat{Z}$ are the **G***-martingale parts of the* **G***-semi martingales m^h and Z, $J_t^h = e^{\Gamma_t}(m_t^h - \int_0^t h_u dF_u)$ and M is the discontinuous* **G***-martingale $M_t = D_t - \Lambda_{t\wedge\tau}$ where $d\Lambda_t = dA_t/G_{t-}$. Furthermore,*

$$J_t^h 1\!\!1_{t<\tau} = H_t 1\!\!1_{t<\tau}\,.$$

Proof: The proof is rather technical and is based on Itô's calculus and property (C). As usual, G^c is the martingale continuous part of the semi-martingale G. We recall that $d[G]_t = d[G^c]_t + (\Delta G_t)^2$. Then, Itô's formula leads to

$$\begin{aligned} d(e^{\Gamma_t}) = d(G_t^{-1}) &= -\frac{1}{(G_{t-})^2}dG_t + \frac{1}{(G_{t-})^3}d[G^c]_t + \left(e^{\Gamma_t} - e^{\Gamma_{t-}} + \frac{1}{(G_{t-})^2}\Delta G_t\right) \\ &= \frac{1}{(G_{t-})^2}\left(-dG_t + \frac{1}{G_{t-}}d[G^c]_t\right) + \frac{1}{G_t\,(G_{t-})^2}(\Delta G_t)^2 \\ &= \frac{1}{(G_{t-})^2}\left(-dG_t + \frac{1}{G_{t-}}d[G^c]_t + \frac{1}{G_t}(\Delta G_t)^2\right)\,. \end{aligned} \tag{6}$$

The quadratic covariation of the processes $(Y_t, t \geq 0)$, where

$$Y_t = m_t^h - \int_0^t h_u dF_u = m_t^h + \int_0^t h_u dG_u$$

and $(e^{\Gamma_t}, t \geq 0)$ is

$$\begin{aligned} d[e^\Gamma, Y]_t &= d[e^\Gamma, m^h]_t + h_t d[e^\Gamma, G]_t \\ &= \frac{1}{(G_{t-})^2}\left[-d[G, m^h]_t + \frac{1}{G_t}(\Delta G_t)^2\Delta m_t^h - h_t d[G]_t + \frac{h_t}{G_t}(\Delta G_t)^3\right] \\ &= \frac{1}{(G_{t-})^2}\left[-d[G, m^h]_t + \frac{1}{G_t}(\Delta G_t)^2\Delta m_t^h - h_t d[G^c]_t - h_t(\Delta G_t)^2 + \frac{h_t}{G_t}(\Delta G_t)^3\right]\,. \end{aligned}$$

From integration by part formula, the dynamics of $J_t^h = Y_t e^{\Gamma_t}$ are

$$\begin{aligned} dJ_t^h &= e^{\Gamma_{t-}}dY_t + Y_{t-}de^{\Gamma_t} + d[e^\Gamma, Y]_t \\ &= -e^{\Gamma_{t-}}(J_{t-}^h - h_t)dG_t + \frac{1}{(G_{t-})^2}(J_{t-}^h - h_t)d[G^c]_t + \frac{1}{G_t G_{t-}}(J_{t-}^h - h_t)(\Delta G_t)^2 \\ &\quad + e^{\Gamma_{t-}}\left(dm_t^h + \frac{1}{G_t G_{t-}}(\Delta G_t)^2\Delta m_t^h\right) - \frac{1}{(G_{t-})^2}d[G, m^h]_t \\ &= e^{\Gamma_{t-}}(J_{t-}^h - h_t)\left(-dG_t + e^{\Gamma_{t-}}d[G^c]_t + \frac{1}{G_t}(\Delta G_t)^2\right) \\ &\quad + e^{\Gamma_{t-}}\left(dm_t^h + \frac{1}{G_t G_{t-}}(\Delta G_t)^2\Delta m_t^h - \frac{1}{G_{t-}}d[G, m^h]_t\right)\,. \end{aligned}$$

The decomposition of F in the filtration **G** is

$$F_{t\wedge\tau} = Z_{t\wedge\tau} + A_{t\wedge\tau} = \widehat{Z}_{t\wedge\tau} - \int_0^{t\wedge\tau}\frac{1}{G_s}d[Z]_s + A_{t\wedge\tau} = 1 - G_{t\wedge\tau}\,.$$

Then, on the set $\{\tau > t\}$

$$dJ_t^h = e^{\Gamma_{t-}}\left[(J_{t-}^h - h_t)d\widehat{Z}_t + d\widehat{m}_t^h + (J_{t-}^h - h_t)dC_t + dK_t\right]$$

where

$$\begin{aligned} dC_t &= \frac{1}{G_t}(\Delta G_t)^2 + e^{\Gamma_{t-}} d[G^c] + dA_t - \frac{1}{G_t} d[Z]_t \\ dK_t &= \frac{1}{G_t G_{t-}} (\Delta G_t)^2 \Delta m_t^h - \frac{1}{G_{t-}} d[G, m^h]_t + \frac{1}{G_t} d[G, m^h]_t . \end{aligned}$$

If G has no jump at time t, $dK_t = 0$, if G has a jump

$$dK_t = -(\Delta G_t)\,(\Delta m_t^h)\,\frac{1}{G_t G_{t-}}\big(-\Delta G_t + G_t - G_{t-}\big) = 0 .$$

The first term is equal to

$$\begin{aligned} dC_t &= \frac{1}{G_t}(\Delta G_t)^2 + e^{\Gamma_t} d[G^c]_t + dA_t - \frac{1}{G_t} d[Z]_t \\ &= \frac{1}{G_t} d[G]_t + dA_t - \frac{1}{G_t} d[Z]_t \\ &= \frac{1}{G_t} d[A]_t + dA_t = \frac{1}{G_t}(\Delta A_t)^2 + dA_t = e^{\Delta \Gamma_t} dA_t \end{aligned}$$

where we have used that, in the case where the $\mathbf{F}$ martingales are continuous $\Delta A = \Delta F$ whereas that A is continuous in the case $P(\tau = \theta) = 0$ (See Jeulin [?] page 65). It remains to check that both members of (??) have the same jump at time τ. The left-hand side jumps from $H_{\tau-}$ to h_τ and the jump of the right-hand side is $e^{\Delta\Gamma_\tau}(h_\tau - J^h_{\tau-})\Delta M_\tau = (h_\tau - H_{\tau-})$. △

Remark 4 In the case where $\mathbf{F}$ is the natural filtration of some Brownian motion W, the $\mathbf{G}$-martingales $\widehat{m}$ and $\widehat{Z}$ can be expressed in terms of the $\mathbf{G}$-martingale part $\widehat{W}$ of the $\mathbf{G}$-semi-martingale W. Obviously, $\widehat{W}$ is a $\mathbf{G}$-Brownian motion. Even in this particular case, it is more difficult to obtain a representation theorem for any $\mathbf{G}$-martingale. In the case where τ an honest time in $\mathcal{F}_\infty$, Azéma et al. [?] have established that any $\mathbf{G}$-martingale can be written as a sum of a stochastic integral with respect to $\widehat{W}$, a stochastic integral with respect to M and a third martingale $v\mathbb{1}_{(\tau\le t)}$ where $v \in \mathcal{F}_\tau^+$ such that $E(v|\mathcal{F}_\tau) = 0$. We recall that $\mathcal{F}_\tau^+$ is generated by the random variables Z_τ where Z is an $\mathbf{F}$-progressively measurable process. For example, if $\tau = \sup\{t \le 1 \,:\, W_t = 0\}$, then $v = V \operatorname{sgn}(W_1)$ with $V \in L^2(\mathcal{F}_\tau)$ (See Yor [?], page 74). This random variable can be viewed as the gap of the information immediatly after default.

Comments 1 Due to hypothesis (C), this theorem does not cover the interesting case where the default time is an unpredictable stopping time in the filtration $\mathbf{F}$, as it is the case in Zhou's paper [?]. In that case, the model would be a structural model and the hedging, if it exists would be the hedging of an $\mathcal{F}_T$-measurable contingent claim as well as its price, in a way very similar to barrier option case.

Corollary 2 *Let $X \in \mathcal{F}_T$ and $X_t = E(X\mathbb{1}_{T<\tau}|\mathcal{G}_t)$. Then,*

$$X_t = m_0^X + \int_0^{t\wedge\tau} e^{\Gamma_{s-}}\left(d\widehat{m}_s^X + e^{\Gamma_{s-}} m_{s-}^X d\widehat{Z}_s\right) - \int_{]0,t\wedge\tau]} e^{\Gamma_s} m_{s-}^X\, dM_s . \tag{7}$$

Here m^X is the $\mathbf{F}$-martingale $m_t^X = E(XG_T|\,\mathcal{F}_t)$.

Proof: The proof follows from Theorem **??** applied to the càg process $h_t = X\mathbb{1}_{T<t}$. △

Corollary 3 *The process L has dynamics given by*

$$dL_t = L_{t-}\left(e^{\Gamma_{t-}} d\widehat{Z}_t - e^{\Delta\Gamma_t} dM_t\right) . \tag{8}$$

Proof: This follows from the previous corollary applied to the martingale $L_t = E(\mathbb{1}_{T<\tau} e^{\Gamma_T}|\mathcal{F}_t)$. △

3.3 Mathematical versus financial examples

Suppose that τ is the last time before maturity where S is at level a, i.e.,

$$\tau = \sup\{t \leq T \,:\, S_t = a\}$$

and $\tau = T$ if the supremum is over an empty set. It is possible to compute in a closed form the hazard process [**?**]. However, if an agent knows when that time occurs, and if the asset S is continuously tradeable up to time T, the agent has obviously an arbitrage opportunity, since he knows that with probability 1, prices will remain below or above that level. Arbitrages occur even on the time interval $[0, \tau[$. Other examples may be found in [**?**]. An open problem is to characterize default time τ such that there are no arbitrages opportunities in the enlarged filtration. We shall give a partial answer while studying (H)-hypothesis, and give in the last section an example where the important question of "life after default" is studied.

3.4 Dynamics of defaultable zero-coupon

Let $\mathcal{P}(\mathcal{G}_T)$ the set of probabilities equivalent to the historical P on the σ-algebra $\mathcal{G}_T$, with a Radon-Nikodym density square integrable. For any $Q \in \mathcal{P}(\mathcal{G}_T)$, we denote by $M(Q, \mathbf{G})$ the set of $Q - \mathbf{G}$ martingales. If the default free zero-coupon $B(\cdot, T)$ and the primary asset S are the only traded assets on the market, the risk associated with the default is not hedgeable, hence the defaultable market is incomplete, and the set of **G**-e.m.m., i.e.

$$\mathcal{M} = \{Q \in \mathcal{P}(\mathcal{G}_T) : SR \in M(Q, \mathbf{G})\}$$

contains an infinite number of probabilities. Let us remark that, if $\mathcal{F}_t^S \subset \mathcal{F}_t$, then the restriction of any **G** e.m.m. to $\mathcal{F}_T$ is an **F**- e.m.m. Indeed, if

$$E_{\tilde{Q}}(R_T S_T | \mathcal{G}_t) = R_t S_t$$

taking the expectation with respect to $\mathcal{F}_t$ of both members, we get

$$E_{\tilde{Q}}(R_T S_T | \mathcal{F}_t) = R_t S_t$$

hence, RS is a $\tilde{Q}|_{\mathcal{F}_T}$ martingale.

We assume now that a defaultable zero-coupon of maturity T (DZC in short) is traded on the market at t-time price $\rho(t, T)$. This DZC delivers one monetary unit at time T if and only if the default has not occur before time T. If the market where the asset S, a default-free zero-coupon and the DZC are traded is arbitrage free, there exists at least one **G**-e.m.m. $\tilde{Q}$ such that the discounted price of the DZC is a **G**-martingale, i.e., $\rho(t, T)R_t = E_{\tilde{Q}}(R_T \mathbb{1}_{T<\tau} | \mathcal{G}_t)$. We emphasize that the e.m.m. is chosen by the market which trades the defaultable zero-coupon at the market price $\rho(t, T)$. We do not assume the uniqueness of e.m.m. $\tilde{Q}$; however, since the DZC is traded, for any e.m.m. $\tilde{Q}$ the equality $\rho(t, T)R_t = E_{\tilde{Q}}(R_T \mathbb{1}_{T<\tau} | \mathcal{G}_t)$ holds. Lemma **??** leads to

$$E_{\tilde{Q}}(R_T \mathbb{1}_{T<\tau} | \mathcal{G}_t) = \mathbb{1}_{t<\tau} e^{\Gamma_t} E_{\tilde{Q}}(R_T G_T | \mathcal{F}_t)$$

where Γ is the $\tilde{Q}$ hazard process, i.e., $e^{-\Gamma_t} = G_t = \tilde{Q}(\tau > t|\mathcal{F}_t)$, or in a concise form $\rho(t,T)R_t = L_t m_t^R$ where m^R is the $\tilde{Q}|_{\mathcal{F}_t}$ martingale $m_t^R = E_{\tilde{Q}}(R_T G_T|\mathcal{F}_t)$.

From (**??**) and $\rho(t,T)R_t = L_t m_t^R$, we deduce

$$R_t\rho(t,T) = m_0^R + \int_0^{t\wedge\tau} e^{\Gamma_{s-}} \left(d\widehat{m}_s^R + \rho(s-,T)R_s d\widehat{Z}_s \right) - \int_{]0,t\wedge\tau]} e^{\Delta\Gamma_s}\rho(s-,T)\, dM_s\,.$$

therefore

$$d\rho(t,T) = \rho(t-,T)\left(r_t dt + e^{\Gamma_{t-}} d\widehat{Z}_t + \frac{1}{m_{t-}^R} d\widehat{m}_t^R - e^{\Delta\Gamma_s}\, dM_s \right)\,.$$

3.5 Hedging strategies

Theorem **??** is quite surprising due to the specific form of the coefficient of M in the second stochastic integral. However, it is quite intuitive in an hedging framework, at least in the case where Γ is continuous. The default arrives by surprise, therefore, the only thing to do in order to insure a portfolio with current value H_t when the value after default will be h_t is to have a long position of $h_t - H_t$ on a fictitious asset who delivers 1 at default time, i.e. D.

4 Particular case : F increasing

In order to avoid enlargement of filtration techniques, we can assume that F is predictable and increasing, hence the predictable increasing process A equals F. In that case, the process $(M_t \overset{def}{=} D_t - \Lambda_{t\wedge\tau}; t \geq 0)$ where $d\Lambda_t = \dfrac{dA_t}{G_{t-}} = \dfrac{dF_t}{G_{t-}}$ is a **G**-martingale.

In the particular case where F is continuous and increasing, $\Lambda = \Gamma$, and the process

$$M = (M_t \overset{def}{=} D_t - \Gamma_{t\wedge\tau}; t \geq 0)$$

is a **G**-martingale. (See e.g. [**?**], [**?**] or [**?**] for direct proofs and comments.)

As a corollary of equation **??**, we obtain

Corollary 4 *Suppose that F is a predictable increasing process. Then the* **G***-martingale $L_t = (1 - D_t)\exp(\Gamma_t)$ satisfies $dL_t = -L_t\, dM_t$.*

Proof: It suffices to note that $L_{t-}e^{\Delta\Gamma_t} = L_t$. △

4.1 Representation theorem

We give in this setting the proof of the representation theorem, already given in [**?**], for **G**-martingales of the form $E(h_\tau|\,\mathcal{G}_t)$ where h is a **F**-predictable bounded process. This theorem is a particular case of the general theorem, however, its proof is really short if we add some regularity conditions. Moreover, we can avoid the condition (C). Bélanger et al. [**?**] made use of the same kind of theorem to hedge contingent claims.

Proposition 1 *Suppose that F is increasing and continuous. Then, the martingale $H_t = E(X\mathbb{1}_{T<\tau}|\,\mathcal{G}_t)$, admits a decomposition in a continuous martingale and a discontinuous martingale as follows :*

$$H_t = m_0^X + \int_0^t L_{u-} dm_u^X - \int_{]0,t]} m_u^X\, e^{\Gamma_u} dM_u, \tag{9}$$

where m^X *is the continuous* **F**-*martingale*

$$m_t^X = E\Big(XG_T|\mathcal{F}_t\Big)\,,$$

$L_t = e^{\Gamma_t}(1-D_t)$ *and* M *is the discontinuous* **G**-*martingale* $M_t = D_t - \Gamma_{t\wedge\tau}$.

Proof: We recall that, if X is a bounded variation process and Y a semi-martingale, the integration by parts formula simplifies and can be written as

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_s dX_s\,.$$

From lemma **??**,

$$E(X1\!\!1_{T<\tau}|\,\mathcal{G}_t) = 1\!\!1_{t<\tau}e^{\Gamma_t}E(XG_T|\,\mathcal{F}_t) = L_t m_t^X\,.$$

Using $dL_t = -L_t dM_t$, and the fact that L is a process with bounded variation, the integration by part leads to

$$E(X1\!\!1_{T<\tau}|\,\mathcal{G}_t) = m_0^X + \int_0^t L_{u-}dm_u^X - \int_{]0,t]} m_u^X e^{\Gamma_u}dM_u\,,$$

on the set $\{t<\tau\}$. The integral $\int_0^t L_{u-}dm_u^X$ is equal to $\int_0^{t\wedge\tau} L_{u-}dm_u^X$ whereas the jump at time τ of $\int_{]0,t]} m_u^X e^{\Gamma_u}dM_u$ is equal to the jump of H. △

Remark 5 In the particular case where G is deterministic and continuous, we get $m_t^X = G_T E_P(X|\mathcal{F}_t)$ and

$$E(X1\!\!1_{T<\tau}|\,\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}G_T\left(m_0^X + \int_0^t e^{\Gamma(s)}dX_s - \int_{]0,t]} e^{\Gamma(s)}X_s dM_s\right),$$

where $X_t = E_P(X|\mathcal{F}_t)$.

4.2 Calibration

If defaultable zero-coupons with different maturities are traded in the market at price $\rho(t,T)$, we can deduce the risk-neutral hazard process from the price of the DZC in the same way that it is possible to deduce the instantaneous interest rate from the price of default free zero coupon, under the assumption that $\rho(t,T)$ is differentiable with respect to the maturity. Indeed, the same proof as in the case of term structure of interest rate establishes that

$$r_t + \gamma_t = -\frac{\partial}{\partial T}\ln\rho(t,T)|_{t=T}$$

where γ stands for the derivative of Γ.

Remark 6 In the general case, we do not know how to deduce the martingale part of F from the knowledge of the price of a DZC.

5 (H) hypothesis

We now study a particular case, where the so-called (H) hypothesis holds, where
(H) *Any* **F** *square integrable martingale is a* **G** *square integrable martingale.*
We discuss the meaning of that hypothesis, its stability under a change of probability measure, its links with arbitrage opportunities in the defaultable world, and we study the hedging of defaultable contingent claims in that setting.

5.1 Characterization of (H) hypothesis

We assume that (H) holds under the probability P, i.e., any P-square integrable $\mathbf{F}$-martingale is a $\mathbf{G}$-martingale. It is well known [**?**] that this is equivalent to

$$\forall\, t, \quad P(\tau \leq t|\mathcal{F}_\infty) = P(\tau \leq t|\mathcal{F}_t). \tag{10}$$

In particular, F and Γ, evaluated under P are increasing processes. This is in particular the case for Cox processes (See e.g. Lando [**?**]) where τ is defined via a given non-negative $\mathbf{F}$-adapted process γ as

$$\tau = \inf\{t \geq 0\,,\, \int_0^t \gamma_s ds \geq \Theta\}$$

where Θ is a given random variable, independent of $\mathbf{F}$, generally chosen with an exponential law.

The following interesting lemma, which establishes that working under (H) hypothesis is equivalent to a Cox process modeling is proved in [**?**].

Lemma 2 *If* (H) *hypothesis holds and F is strictly increasing and continuous, then the random variable Γ_τ is exponentially distributed and independent of $\mathcal{F}_\infty$. Hence,*

$$\tau = \inf\{t\,:\,\Gamma_t \geq \Theta\}$$

where Θ is an exponential random variable, independent of $\mathcal{F}_\infty$.

Proof: Let us reproduce here the proof. We suppose that (H) holds, which implies that

$$P(\tau \leq t|\mathcal{F}_\infty) = e^{-\Gamma_t}\,.$$

Setting $\Theta \overset{def}{=} \Gamma_\tau$, leads to

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$P(\Theta > u|\mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established that Θ is an exponential random variable, independent of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$. △

5.2 Brownian filtration, stability of (H) hypothesis

Let us now assume that (H) hypothesis holds under P and that $\mathbf{F}$ is a Brownian filtration generated by the Brownian motion W. In that setting, Kusuoka [**?**] establishes, when F is continuous the following representation theorem :

Theorem 2 *Assume that* (H) *hypothesis holds under P and that* $\mathbf{F}$ *is a Brownian filtration. Then, any* $\mathbf{G}$*-square integrable martingale admits a representation as a sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M.*

This theorem admits a straightforward extension to the case where F is discontinuous. Furthermore, it allows us to characterize the set of e.m.m.

Let Q be any probability equivalent to P on the filtration $\mathbf{G}$. The Radon-Nikodym density ζ of Q with respect to P is a strictly positive $\mathbf{G}$ martingale. If ζ is square integrable, using Kusuoka's theorem, it can be written as

$$d\zeta_t = \zeta_{t-}(\psi_t dW_t + \phi_t dM_t),\ \zeta_0 = 1\,,$$

where $\phi_t > -1$. Using Doléans-Dade's exponential, we write in a concise form

$$\zeta_t = \mathcal{E}(\psi W)_t\,\mathcal{E}(\phi M)_t\,.$$

Let us restrict our attention to the case where ψ is $\mathbf{F}$-adapted.

Proposition 2 *Let $\zeta_t = \mathcal{E}(\psi W)_t\,\mathcal{E}(\phi M)_t$ where ψ is $\mathbf{F}$-adapted. If* (H) *holds under P, then it holds also under Q where $dQ|_{\mathcal{G}_t} = \zeta_t dP|_{\mathcal{G}_t}$.*

Proof: Let Q^* be defined on $\mathcal{G}_t$ by $dQ^* = \mathcal{E}(\phi M)_t dP$. From Girsanov's theorem, the P-$\mathbf{G}$ Brownian motion W is a Q^*-$\mathbf{G}$ Brownian motion therefore a Q^*-$\mathbf{F}$ Brownian motion. Any Q^*-$\mathbf{F}$ martingale can be written as a stochastic integral with respect to W, and (H) holds under Q^*.
Since $\mathcal{E}(\psi W)_t$ is $\mathcal{F}_t$-adapted, for any $t < T$,

$$Q(\tau \le t|\mathcal{F}_T) = \frac{E_P(\mathbb{1}_{\{\tau\le t\}}\,\zeta_T|\mathcal{F}_T)}{E_P(\zeta_T|\mathcal{F}_T)} = \frac{E_P(\mathbb{1}_{\{\tau\le t\}}\,\mathcal{E}(\phi M)_T|\mathcal{F}_T)}{E_P(\mathcal{E}(\phi M)_T|\mathcal{F}_T)} = Q^*(\tau \le t|\mathcal{F}_T)\,.$$

Since (H) holds under Q^*, we obtain

$$Q(\tau \le t|\mathcal{F}_T) = Q^*(\tau \le t|\mathcal{F}_t) = Q(\tau \le t|\mathcal{F}_t)$$

and (H) holds under Q. △

A particular case is when the underlying asset follows

$$dS_t = S_t(\alpha_t dt + \sigma_t dW_t)$$

where α and σ are $\mathbf{F} = \mathbf{F}^W$-adapted and where the e.m.m. for the filtration $\mathbf{F}$ is unique. The set of equivalent $\mathbf{G}$-martingale measures is characterized by the set of Radon-Nikodym densities ζ^ϕ of the form

$$d\zeta^\phi_t = \zeta^\phi_{t-}(\theta_t dW_t + \phi_t dM_t) \tag{11}$$

where $\theta_t = \dfrac{\alpha_t - r_t}{\sigma_t}$ is $\mathbf{F}$-adapted and ϕ is any $\mathbf{G}$-adapted process, such that $\phi > -1$. In this case, hypothesis (H) holds under any e.m.m.. Moreover, the restriction to $\mathbf{F}$ of any e.m.m. for the $\mathbf{G}$ filtration is equal to the e.m.m. for the filtration $\mathbf{F}$.

Comments 2 In general, (H) hypothesis is not stable under a change of probability. See Kusuoka [**?**] for a counterexample.

Remark 7 When Γ is absolutely continuous with respect to Lebesgue measure, i.e., $\Gamma_t = \int_0^t \gamma_s ds$, it is easy to prove that there exists a probability Q^*, equivalent to P such that under Q^*, the random variable τ is independent of $\mathcal{F}_\infty$. Indeed, the change of probability $dQ^*|_{\mathcal{G}_t} = \zeta_t dP|_{\mathcal{G}_t}$, where $d\zeta_t = \zeta_t \varphi_t dM_t$, and $\varphi_t = (\gamma_t)^{-1} - 1$ leads to $Q^*(\tau \le t|\mathcal{F}_\infty) = 1 - e^{-t}$ and the independence follows. Hence, W is a Q^*-Brownian motion in the filtration $\mathcal{G}_t$. Furthermore, the process M^* where $dM^*_t = dM_t - \zeta_t\varphi_t\gamma_t dt = dD_t - (1 - D_t)dt$ is a Q^*-martingale, independent of W. Therefore, under Q^* any $\mathbf{G}$ martingale is the sum of a stochastic integral with respect to W and a stochastic integral with respect to M^*, therefore the predictable representation holds also under P. This argument yields to a simple proof of Kusuoka's result.

5.3 Range of prices

Suppose that the tradeable assets are the riskless one and the risky asset S. Then, the defaultable market is incomplete (there are no assets to hedge the default risk) and the set of **G**-e.m.m. is infinite and characterized by mean of the process ϕ (see (**??**)). The law of the default time depends strongly on the choice of an e.m.m. Indeed, using Girsanov's theorem, if the hazard rate of τ is $\Gamma_t = \int_0^t \gamma_s ds$ under P, and $dQ = \zeta_t^\phi dP$, the hazard rate process of τ is $\Gamma_t^\phi = \int_0^t \gamma_s(1+\phi_s)ds$ under Q. In particular, the range of prices for the payoff $X 1\!\!1_{T<\tau}$ is a large interval whose bounds correspond to the case where $\tau = 0$ and $\tau = \infty$, i.e. $]0, E_Q(X)[$. Note that the value of $E_Q(X)$ does not depend on the choice of Q. It suffices to choose a sequence of processes ϕ such that, in one hand, $\phi_t^n \to \infty$ and in the other hand $\phi_t^n \to -1$.

5.4 Arbitrage

We discuss now the hypothesis on the modeling of default time that we require in order to avoid arbitrages in the defaultable market. We return for a while to the general modelling of section **??**.

Proposition 3 *Let S be a semi-martingale on $(\Omega, \mathcal{G}, P)$ such that there exists a unique probability Q, equivalent to P on $\mathcal{F}_T$, where $\mathcal{F}_t = \mathcal{F}_t^S = \sigma(S_s, s \leq t)$ such that $(\widetilde{S}_t = S_t R_t, 0 \leq t \leq T)$ is an $\mathbf{F}^S$-martingale under the probability Q. We assume that there exists a probability $\tilde{Q}$, equivalent to P on $\mathcal{G}_T$ such that $(\widetilde{S}_t = S_t R_t, 0 \leq t \leq T)$ is a $\mathbf{G}$-martingale under the probability $\tilde{Q}$. Then,* (H) *holds under Q and the restriction of $\tilde{Q}$ to $\mathcal{F}_T$ is equal Q.*

PROOF: This is quite obvious. We give a "financial proof". Under the hypothesis, any square integrable $\mathbf{F} - Q$ martingale can be thought as the discounted value of a contingent claim $\xi \in \mathcal{F}_T$. Since the same claim exists in the larger market, which is assumed to be arbitrage free, the claim process is also a $\mathbf{G} - \tilde{Q}$ martingale[2]. From the uniqueness of price for hedgeable claims, for any contingent claim $X \in \mathcal{F}_T$ and any **G**-e.m.m. $\tilde{Q}$,

$$E_Q(XR_T|\mathcal{F}_t) = E_{\tilde{Q}}(XR_T|\mathcal{G}_t)\,.$$

In particular, $E_Q(Z) = E_{\tilde{Q}}(Z)$ for any $Z \in \mathcal{F}_T$ (take $t = 0$ and $X = ZR_T^{-1}$), hence the restriction of any e.m.m. $\tilde{Q}$ to the σ-algebra $\mathcal{F}_T$ equals Q. Moreover, since any square integrable $\mathbf{F}$-Q-martingale can be written as $E_Q(X|\mathcal{F}_t) = E_{\tilde{Q}}(X|\mathcal{G}_t)$, we get that any square integrable $\mathbf{F}$-$\tilde{Q}$-martingale is a $\mathbf{G}$-$\tilde{Q}$-martingale. △

Comments 3 In the literature, it is generally assumed that the defaultable market is complete and arbitrage free. If this assumption means that the set of contingent claims is the set of $\mathcal{G}_T$-measurable random variable, then, in particular, any $\mathcal{F}_T$-measurable random variable is a tradeable contingent claim, and S_T is a tradeable asset.

5.5 Hedging strategies

We suppose that the default-free market including the default-free zero-coupon and the risky asset S is complete and arbitrage free, we denote by Q the $\mathcal{F}_T$-e.m.m. For any process V, we denote by $\tilde{V}$ the

[2]We thank the referee for writing a more concise proof that our original one.

discounted value of V, i.e., $\tilde{V}_t = R_t V_t$. We assume that a defaultable zero-coupon is available on the market and that (H) holds under Q. We also assume that the process F is continuous.

We denote by $\tilde{Q}$ a $\mathcal{G}_T$-e.m.m. We recall that $\tilde{Q}$ and Q are equal on the σ-algebra $\mathcal{F}_T$. We now make precise the hedging of a defaultable claim and check that the market, including the DZC is complete.

We recall that a pair (a, v) of $\mathbf{F}$-adapted processes is an hedging strategy for the contingent claim $V_T \in \mathcal{F}_T$ if, denoting by $V_t = a_t S^0_t + v_t S_t$ the t-time value of this strategy, the self-financing relation $dV_t = a_t dS^0_t + v_t dS_t$ holds and $V_T = a_T S^0_T + v_T S_T$. A self-financing strategy is characterized by its initial value x and the process v via $R_t V_t = x + \int_0^t v_s d\tilde{S}_s$. The number of shares of riskless asset for this strategy is $\alpha_t = \tilde{V}_t - v_t \tilde{S}_t$. The process v describes the number of shares of the asset held in the self-financing strategy.
In the same way, if the risky asset S and a DZC $\rho(\cdot, T)$ with maturity T are traded, a self-financing strategy is a triple (a, b, c) of $\mathbf{G}$-adapted processes such that if $Z_t = a_t S^0_t + b_t S_t + c_t \rho(t, T)$ is the value of this strategy, the self-financing relation $dZ_t = a_t dS^0_t + b_t dS_t + c_t d\rho(t, T)$ holds and $Z_T = a_T S^0_T + b_T S_T + c_T \rho(T, T)$.

5.5.1 Terminal payoff

We study in a first step the case where there is no rebate, i.e., we consider terminal payoff of the form $X \mathbb{1}_{T<\tau}$. We compute $E_{\tilde{Q}}(X \mathbb{1}_{T<\tau} R_T | \mathcal{G}_t)$ by mean of our representation theorem, and we give the hedging strategy for $X \mathbb{1}_{T<\tau}$ based on riskless asset, risky asset and defaultable zero-coupon.

Theorem 3 *The hedging strategy (a, b, c) for the defaultable contingent claim $X \mathbb{1}_{T<\tau}$, based on the riskless bond, the asset and the defaultable zero-coupon satisfies $c_t \rho(t, T) = R_t^{-1} E_{\tilde{Q}}(X R_T \mathbb{1}_{T<\tau} | \mathcal{G}_t)$. Hence, $a_t S^0_t + b_t S_t = 0$.*
More precisely, let $(\tilde{V}^X_t - v^X_t \tilde{S}_t, v^X_t)$ be the hedging strategy for the default-free contingent claim $X G_T$, and $(\tilde{V}_t - v_t \tilde{S}_t, v_t)$ the hedging strategy for the default-free contingent claim G_T. Then,

(i) $c_t = \dfrac{V^X_t}{V_t}$

(ii) $b_t = e^{\Gamma_t}(v^X_t - \dfrac{V^X_t}{V_t} v_t)$

(iii) $a_t = -e^{\Gamma_t}(v^X_t - \dfrac{V^X_t}{V_t} v_t)\tilde{S}_t$.

Proof: Let $G_t = \tilde{Q}(t < \tau | \mathcal{F}_t)$ where $\tilde{Q}$ is the $\mathbf{G}$ e.m.m.. Since any contingent claim in $\mathcal{F}_T$ is hedgeable in the default-free market, there exists a predictable process $(v^X_t, t \geq 0)$ and a constant x such that

$$X G_T R_T = x + \int_0^T v^X_s d\tilde{S}_s \,.$$

Hence, the t-time price V^X_t of $X G_T$ is given via

$$\tilde{V}^X_t = V^X_t R_t = E_{\tilde{Q}}(X G_T R_T | \mathcal{F}_t) = x + \int_0^t v^X_s d\tilde{S}_s \,.$$

The strategy $(\tilde{V}^X_t - v^X_t \tilde{S}_t, v^X_t)$ is now a self-financing strategy hedging the contingent claim $X G_T$.
From the representation theorem applied to

$$\tilde{H}_t = E_{\tilde{Q}}(X R_T \mathbb{1}_{T<\tau} | \mathcal{G}_t) = (1 - D_t) e^{\Gamma_t} E_Q(X R_T G_T | \mathcal{F}_t) = L_t \tilde{V}^X_t \,,$$

we obtain

$$XR_T 1\!\!1_{T<\tau} = V_0^X + \int_0^{T\wedge\tau} e^{\Gamma_u} d\tilde{V}_u^X - \int_{]0,T\wedge\tau]} \tilde{V}_u^X e^{\Gamma_u} dM_u \,.$$

Let us denote $\tilde{V}_t = E_Q(R_T G_T|\mathcal{F}_t) = \tilde{V}_t^1$ the discounted price of G_T and $v_t = v_t^1$ its hedging portfolio. Then, using that $d\tilde{\rho}_t - L_{t-}d\tilde{V}_t = -L_t\tilde{V}_t dM_t$,

$$\begin{aligned} XR_T 1\!\!1_{T<\tau} &= V_0^X + \int_0^{T\wedge\tau} e^{\Gamma_s}(v_s^X - \frac{\tilde{V}_s^X}{\tilde{V}_s} v_s) d\tilde{S}_s + \int_{]0,T\wedge\tau]} \frac{\tilde{V}_s^X}{\tilde{V}_s} d\tilde{\rho}_s \\ &= V_0^X + \int_0^{T\wedge\tau} e^{\Gamma_s}(v_s^X - \frac{V_s^X}{V_s} v_s) d\tilde{S}_s + \int_{]0,T\wedge\tau]} \frac{V_s^X}{V_s} d\tilde{\rho}_s \,. \end{aligned} \quad (12)$$

Now, using that $\rho_t = L_t V_t$, the position on the riskless asset is the value H_t of the portfolio minus the wealth invested in the risky securities, i.e.

$$L_t V_t^X - \frac{V_t^X}{V_t}\rho_t - e^{\Gamma_t}(v_t^X - \frac{V_t^X}{V_t} v_t) S_t = -e^{\Gamma_t}(v_t^X - \frac{V_t^X}{V_t} v_t) S_t \,.$$

From the construction, this strategy is self-financing. The amount of money invested in the DZC is

$$\frac{V_t^X}{V_t}\rho(t,T) = L_t V_t^X = H_t$$

that is exactly the opposite of the loss that can occurs for immediate default. △

Remark 8 It is easy to check that the given pair (a,b) is the unique pair (a,b) such that

$$(a, b, \, (\rho(t,T)R_t)^{-1} E_Q(XR_T 1\!\!1_{T<\tau}|\mathcal{G}_t))$$

is self-financing.

Remark 9 In the particular case where Γ is deterministic, we get immediatly

$$XR_T 1\!\!1_{T<\tau} = h + \int_0^{T\wedge\tau} e^{\Gamma_s} v_s^X d\tilde{S}_s + \int_{]0,T\wedge\tau]} E_Q(XR_T|\mathcal{F}_s) d\tilde{\rho}_s \,.$$

5.5.2 Rebate part

The representation theorem provides also an hedging strategy for a rebate h paid at hit.

Proposition 4 *Let (a,b,c) be the hedging strategy for the rebate part, i.e., the self-financing strategy such that*

$$R_t(a_t S_t^0 + b_t S_t + c_t\rho(t,T)) = E_{\tilde{Q}}(h_\tau 1\!\!1_{\tau\le T} R_\tau|\mathcal{G}_t) \,.$$

Then, $c_t\rho(t,T) = R_t^{-1} E_{\tilde{Q}}(h_\tau 1\!\!1_{\tau\le T} R_\tau|\mathcal{G}_t) - h_t$. More precisely, the hedging strategy before default time of the rebate part, paid at hit, consists of

(i) $c_t = \frac{1}{V_t}(C_t^h - e^{-\Gamma_t} h_t)$

(ii) $b_t = e^{\Gamma_t}[v_t^h - \frac{1}{V_t} v_t C_t^h] + \frac{1}{V_t} v_t h_t$

(iii) $a_t S_t^0 = S_t[\frac{1}{V_t} v_t h_t - e^{\Gamma_t}[v_t^h + \frac{1}{V_t} v_t C_t^h]] + h_t,$

where $C_t^h = (R_t)^{-1} E_Q(\int_t^T h_u R_u dF_u|\mathcal{F}_t)$.

We compute the quantity $E_{\tilde{Q}}(h_\tau \mathbb{1}_{\tau\leq T} R_\tau|\mathcal{G}_t)$, which corresponds to the price of the rebate, when the compensation is payed at hit.

We denote by C^h_t the price of the contingent claim which consists of a dividend hdF paid between time t and T, i.e.

$$\tilde{C}^h_t = E_Q(\int_t^T R_u f_u h_u du|\mathcal{F}_t)\,,$$

and by v^h the associated hedging strategy in the default-free world, i.e.,

$$\tilde{V}^h_t = \tilde{C}^h_t + \int_0^t R_u h_u dF_u = C^h_0 + \int_0^t v^h_s d\tilde{S}_s\,.$$

The representation theorem states that

$$E_{\tilde{Q}}(h_\tau R_\tau \mathbb{1}_{\tau\leq T}|\mathcal{G}_t) = C^h_0 + \int_0^{t\wedge\tau} e^{\Gamma_u} V^h_u d\tilde{S}_u + \int_{[0,t\wedge\tau[}(h_u R_u - J_{u-})dM_u\,,$$

where, on the set $t<\tau$,

$$J_t = E_{\tilde{Q}}(h_\tau R_\tau \mathbb{1}_{\tau\leq T}|\mathcal{G}_t) = e^{\Gamma_t} E_Q(\int_t^T h_u R_u dF_u|\mathcal{F}_t) = e^{\Gamma_t}\tilde{C}^h_t\,.$$

Hence, introducing $\tilde{V}_t$, the discounted price of G_T

$$E_{\tilde{Q}}\left(h_\tau R_\tau \mathbb{1}_{\tau\leq T}|\mathcal{G}_t\right) = C^h_0 + \int_0^{t\wedge\tau} e^{\Gamma_u} v^h_u d\tilde{S}_u - \int_{[0,t\wedge\tau[}(h_u R_u - e^{\Gamma_u}\tilde{C}^h_u)\frac{1}{L_u\tilde{V}_u}[d\tilde{\rho}_u - L_u d\tilde{V}_u]$$

which leads to

$$E_{\tilde{Q}}\left(h_\tau R_\tau \mathbb{1}_{\tau\leq T}|\mathcal{G}_t\right) \quad = \quad V^h_0 + \int_0^{t\wedge\tau}\left[e^{\Gamma_u}\left(v^h_u - C^h_u\frac{v^h_u}{V_u}\right) + \frac{v_u h_u}{V_u}\right]d\tilde{S}_u - \int_{[0,t\wedge\tau[}(h_u - e^{\Gamma_u}C^h_u)\frac{1}{L_u V_u}d\tilde{\rho}_u$$

△

Corollary 5 *Under* (H) *hypothesis, the defaultable-market is complete as soon as a defaultable zero-coupon is traded.*

6 Further research

6.1 Behaviour after default time

Suppose now that the asset S is no more traded after default time (this is the case for houses after an earthquake or an hurican). In that case, the tradeable asset is $S^*_t = S_t \mathbb{1}_{t<\tau}$. Assume that there exists an e.m.m. Q^* such that the (discounted) asset S^* is a $Q^* - \mathbf{G}$ martingale. Then, in the case where $r=0$, $E_{Q^*}(S^*_t|\mathcal{G}_s) = S^*_s$. From lemma **??** applied to the probability Q^*, we have

$$E_{Q^*}(S^*_t|\mathcal{G}_s) = \mathbb{1}_{s<\tau} e^{\Gamma_s} E_{Q^*}(S_t G_t|\mathcal{F}_s)$$

where $e^{-\Gamma_s} = G_s = Q^*(\tau > s|\mathcal{F}_s)$. Taking the expectation with respect to $\mathcal{F}_s$, we get in the one hand

$$E_{Q^*}(S^*_t|\mathcal{F}_s) = E_{Q^*}(S^*_s|\mathcal{F}_s) = S_s E_{Q^*}(\mathbb{1}_{s<\tau}|\mathcal{F}_s) = S_s G_s$$

and in the other hand

$$E_{Q^*}(S^*_t|\mathcal{F}_s) = e^{\Gamma_s} E_{Q^*}(S_t G_t|\mathcal{F}_s) Q^*(s<\tau|\mathcal{F}_s) = E_{Q^*}(S_t G_t|\mathcal{F}_s)$$

hence

$$S_s G_s = E_{Q^*}(S_t G_t|\mathcal{F}_s)$$

and the restriction of the e.m.m. Q^* to the filtration $\mathbf{F}$ is not an e.m.m..