Optimal Design of the Guarantee for Defined Contribution Funds.

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Abstract: In an affine term structure of the interest rates framework, we consider the case of a defined contribution pension fund in the presence of a minimum guarantee. The question we solve is the design of the minimum guarantee in order to minimize the randomness of the benefit. Assuming that the pension fund's retribution is equal to a fixed part of the surplus (that is the difference between the final value of the portfolio managed by the pension fund and the guarantee), we show in this paper that the optimal benefit consists always in a non-random guarantee and a certain fixed part of the surplus. As a by-product, we state a market efficiency test: the variance of the surplus must be proportional to the square of the expectation. These results are based on the self-financing characteristic of an auxiliary process.

Key words: pension funds, minimum guarantee, stochastic interest rates, stochastic optimization.

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1 Introduction

There exist two types of radically different pension funds methods: "defined benefit" method where the contributions are the random variables, and "defined contribution" method where randomness comes from the benefit. Historically, pension funds used mainly the first method which is preferred by the client (see e.g. Davis 1995). However, due to the demographic evolution and the development of the equity markets, new systems have been invented. Nowadays, the pension funds propose mainly defined contribution schemes which transfer the equity market risk to the clients.

A simple way to moderate this inconvenience for the clients, is to introduce a minimum guarantee on the future benefit that will be paid out to the clients. However, this guarantee can be very complex and the question is to find the optimal form that it should take in order to minimize the randomness of the benefit.

In an equity and bond markets framework and assuming that the pension fund's retribution is equal to a fixed part of the surplus (that is the difference between the final value of the portfolio managed by the pension fund and the guarantee), we show in this paper that the optimal benefit consists always in a non-random guarantee and a certain fixed part of the surplus. Moreover, we state a market efficiency test: the variance of the surplus turns out to be proportional to the square of the expectation. This comes from a decreasing quadratic relation between the variance of the surplus and the minimum guarantee.

The manager of the pension fund will invest the wealth in order to optimize the expected value of utility of its share of the surplus. We suppose that the fund manager can continuously adapt its investment strategy in a financial market containing cash, a zero-coupon bond and a stock and where the interest rates follow an affine term structure model with constant parameters, which includes as special cases the Vasicek (1977) and the Cox-Ingersoll-Ross (1985) models. Having proved that an auxiliary process is self-financed (which is the key point of the paper), we determine the analytic form of the guarantee which minimizes the variance of the benefit. Using the martingale method introduced by Cox and Huang (1990, 1991) and Karatzas et al. (1987, 1989) like in Deelstra, Grasselli and Koehl (2000) in a Cox-Ingersoll-Ross framework, it is possible to analyze the investment strategy of the fund manager.

In related literature, Boulier, Huang and Taillard (1999) study the optimal management of a defined contribution plan where the guarantee depends on the level of interest rates at the fixed retirement date. Jensen and Sørensen (1999) measure the effect of a minimum interest rate guarantee constraint through the wealth equivalent in case of no constraints and show numerically that guarantees may induce a significant utility loss for relatively risk tolerant investors. Both the papers by Boulier, Huang and Taillard (1999) and Jensen and Sørensen (2000) choose a Vasiček specification of the term structure in the spirit of Bajeux-Besnainou, Jordan and Portait (1998, 1999).

The paper is organized as follows: in section 2 we define the market structure and introduce the optimization problems. In section 3, we obtain the main property of the auxiliary process. In section 4, we derive the optimal form of the guarantee. Section 5 concludes the paper.

2 The model

In this section, we describe the financial market and the optimization programs.

2.1 The financial market

Randomness is described by a 2-dimensional Brownian motion

$$\underline{z}(t) = \left\{ (z(t), z_r(t))', \ t \in [0, +\infty[\right\}$$

defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where P is the real world probability and $'$ denotes transposition. The filtration $F = (\mathcal{F}_t)_{t \geq 0}$, represents the information structure generated by the Brownian motion and is assumed to satisfy the usual conditions.

Hereafter E_t stands for $E(. \mid \mathcal{F}_t)$, the conditional expected value under the real world probability.

The market is composed of three financial assets, that the agent can buy or sell continuously without incurring any restriction as short sales constraints or any trading cost.

The first asset is the riskless asset (i.e. the cash). Its price, denoted by $S_0(t), t \geq 0$, evolves according to:

$$\frac{dS_0(t)}{S_0(t)} = r_t dt; \quad S_0(0) = 1;$$

where the dynamics of the short rate process r_t are described by the following stochastic differential equation:

$$dr_t = (a - br_t)dt - \sqrt{\eta_1 r_t + \eta_2}\, dz_r(t); \quad t \geq 0; \tag{1}$$

$r_0; a; b; \eta_1$ and η_2 being positive constants.

These dynamics have been studied by Duffie and Kan (1996). Their paper shows that, under these dynamics, the term structure of the interest rates is affine. Moreover, the converse is true under a regularity hypothesis.

Note that these dynamics recover, as special cases, the Vasicek (1977) (resp. Cox-Ingersoll-Ross 1985) dynamics, when η_1 (resp. η_2) is equal to zero.

The second asset is the stock, whose price is denoted by $S(t); t \geq 0$: The dynamics of $S(t)$ are given by:

$$\frac{dS(t)}{S(t)} = r_t dt + \sigma_1 (dz(t) + \lambda_1 dt) + \sigma_2 \sqrt{\eta_1 r_t + \eta_2}\left(dz_r(t) + \lambda_2 \sqrt{\eta_1 r_t + \eta_2}\, dt \right); \tag{2}$$

with $S(0) = 1$ and $\lambda_1; \lambda_2$ (resp. $\sigma_1; \sigma_2$) being constant (resp. positive constants).

The third asset is a zero-coupon bond with maturity T, whose price at time t is denoted by $B(t;T); t \geq 0$.

The following proposition fixes the dynamics of the zero-coupon bond price.

Proposition 1 Let us denote $B(t;T)$ the price at date t of the zero coupon bond maturing at date T. Then:

$$\frac{dB(t;T)}{B(t;T)} = r_t dt + \sigma_B(T - t; r_t)\left(dz_r(t) + \lambda_2 \sqrt{\eta_1 r_t + \eta_2}\, dt \right); \quad B(T;T) = 1 \tag{3}$$

where

$$\sigma_B(T - t; r_t) = h(T - t)\sqrt{\eta_1 r_t + \eta_2}$$

with

$$h(t) = \frac{2(e^{\delta t} - 1)}{\delta - (b - \eta_1 \lambda_2) + e^{\delta t}(\delta + b - \eta_1 \lambda_2)}; \quad t \geq 0; \tag{4}$$

$$\delta = \sqrt{(b - \eta_1 \lambda_2)^2 + 2\eta_1}.$$

Proof. See e.g. Musiela and Rutkowski (1997) section 12.3. ■

At last, we assume that the parameters are such that the financial market is arbitrage-free and complete. Then, for any $t \geq 0$, we can define the deflator price process

$$H(t) = \exp\left\{ -\int_0^t r_s ds - \int_0^t \theta'(s) dz(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds \right\}, \qquad (5)$$

with $\theta'(s) = (\lambda_1, \lambda_2 \sqrt{\gamma_1 r_s + \gamma_2})$.

2.2 The optimization program of the contributor

The contributor pays a flow to the pension fund. This flow consists in a lump sum at date 0, denoted by W_0, and a continuously payed premium, at a rate denoted by $c(t)$, $t \in [0, T]$; the flow of contributions is assumed to be a non-negative, progressive measurable and square-integrable process. Then the value at date 0 of the cash given by the contributor to the pension fund is equal to:

$$W_0^0 = W_0 + E\left[\int_0^T H(s) c_s ds\right].$$

In exchange, the fund manager will provide at date T a benefit which consists of two parts: The first part G_T is guaranteed, which means that the benefit will be greater than G_T almost surely. The second part is a fixed fraction of the surplus Y_T (G_T) (the difference between the terminal wealth W_T of the managed portfolio and the guarantee G_T).

Let us denote by β the fixed fraction of the surplus that will be kept by the fund manager, as a way to incite him. The total benefit of the contributor at date T is then equal to:

$$G_T + (1 - \beta)(W_T - G_T).$$

For technical reasons, we assume that G_T is a strictly positive F_T measurable random variable which is L^p integrable with $p > 2$.

The problem of the contributor is to choose the best contract between those offered by the pension funds, everything else fixed - that is the value of the cash given by the contributor W_0^0, the fraction β of the surplus kept by the fund manager, and its risk aversion that we introduce more in details in the next section. The guarantee is then the only remaining variable and the problem is to find its optimal form. This problem is a static one from the contributor point of view since he has a decision to make at date 0 only for a benefit that will be delivered at date T.

At this stage we assume that the objective function of the contributor depends on the two following variables: the variance of the benefit and the market value at date 0 of the benefit. Moreover, we assume also that the objective function is decreasing (respectively increasing) with respect to the first (second) variable. In the following, we will use the obvious remark that the optimal guarantee G_T^* must lie in the set $\mathcal{G} = \left\{ G_T^k \mid \text{ for } k \in \mathbb{R}^+; G_T^k \text{ is solution of (6)} \right\}$, where (6) is the following optimization program:

$$\max_{G_T} E\left[H_T\left((1-\beta)Y_T(G_T)+G_T\right)\right] \tag{6}$$

under the constraint:

$$Var\left[(1-\beta)Y_T(G_T)+G_T\right]=k$$

It is important to remark that a classical approach should be to consider that the objective function depends on (i) the variance of the benefit and (ii) the expected value of the benefit under the true probability P. However, let us assume that this is the case, and let us consider the associated auxiliary program

$$\max_{G_T} E\left[(1-\beta)Y_T(G_T)+G_T\right] \tag{7}$$

under the constraint:

$$Var\left[(1-\beta)Y_T(G_T)+G_T\right]=k$$

defined in a similar way as above for a fixed k. Denoting by G_T^* and $\overline{G}_T$ the solutions of (6) and (7) respectively, we have :

$$\begin{aligned}
E\left[(1-\beta)Y_T(G_T^*)+G_T^*\right] &\leq E\left[(1-\beta)Y_T(\overline{G}_T)+\overline{G}_T\right]\\
E\left[H_T\left((1-\beta)Y_T(G_T^*)+G_T^*\right)\right] &\geq E\left[H_T\left((1-\beta)Y_T(\overline{G}_T)+\overline{G}_T\right)\right]\\
Var\left[(1-\beta)Y_T(G_T^*)+G_T^*\right] &= Var\left[(1-\beta)Y_T(\overline{G}_T)+\overline{G}_T\right]=k
\end{aligned}$$

and, even if the agent prefers $\overline{G}_T$, it is more efficient for him to receive G_T^* first, then to sell it on the market and to buy $\overline{G}_T$, making a gain of

$$E\left[H_T\left((1-\beta)\left(Y_T(G_T^*) - Y_T(\overline{G}_T)\right) + G_T^* - \overline{G}_T\right)\right] \geq 0.$$

This is the reason why we prefer to maximize the market value of the benefit instead of the expected value of the benefit under the true probability.

At last, note that we can consider the auxiliary programs that minimize the variance under a constraint upon the value of the benefit, which are equivalent to the programs considered in (6).

Formally, these programs are written as follows:

$$\min_{G_T} Var\left[(1-\beta)Y_T(G_T) + G_T\right] \tag{8}$$

under the constraint:

$$E[H(T)((1-\beta)Y_T(G_T) + G_T)] = k$$

In order to solve this problem, we need to look more deeply at the way used by the pension fund to manage the portfolio, in order to get the principal features of $Y_T(G_T)$.

2.3 The optimization program of the pension fund manager

In this subsection, we describe the portfolio problem faced by the pension fund manager. More precisely, we assume:

(i) that the risk aversion of the fund manager is described by a power utility function

$$U(y) = \frac{y^\gamma}{\gamma}; \quad \gamma \in (-\infty; 1)\setminus\{0\}; \tag{9}$$

(ii) that he maximizes the expected utility of his terminal wealth (that is, his part of the surplus).

An investment strategy $\underline{u}_t = \left\{\left(1-u_t^B-u_t^S; u_t^B; u_t^S\right)'\right\}_{0\le t\le T}$ is a $\mathcal{F}$ adapted process, where $\left(1-u_t^B-u_t^S\right)$; (resp. u_t^B; resp. u_t^S) denotes the proportion of the wealth invested into the riskless asset (resp. the bond, resp. the stock) at date t.

Denoting by $W(t)$ the wealth of the fund at date $t \in [0,T]$, his optimization program will be:

$$\max_{(\underline{u}_t)_{t \in [0,T]} \in \mathcal{A}^W} \frac{1}{\gamma} E\left(W(T) - G_T\right)^{\gamma} \tag{10}$$

under the constraints:

$$dW(t) = W(t)\underline{u}_t' \operatorname{diag}[\underline{S}(t)]^{-1} d\underline{S}(t) + c_t dt \tag{11}$$

with:

$$\begin{aligned} & W(0) = W_0 > 0 \\ & \underline{S}(t) = (S_0(t), B(t,T), S(t))' \\ & \underline{u}_t = \left(1 - u_t^B - u_t^S, u_t^B, u_t^S\right)' \end{aligned}$$

and

$$\mathcal{A}^W = \left\{ \begin{array}{l} \left(\underline{u}_t = \left(1 - u_t^B - u_t^S, u_t^B, u_t^S\right)'\right)_{0 \le t \le T} \ \mathcal{F} \text{ adapted process such that:} \\ \int_0^T \left(\left(W(t)u_t^B h(T-t)\right)^2 + \left(W(t)u_t^S\right)^2 \right) dt < +\infty \quad P-a.e.; \\ W(T) - G_T \ge 0 \quad P-a.e. \end{array} \right\}.$$

From now on, we assume that:

$$E[H(T)G_T] < W_0 + E\left[\int_0^T H(t)c_t dt\right] = W_0'; \tag{12}$$

which is equivalent to say that the set of admissible strategies $\mathcal{A}^W$ is non empty. Indeed, by investing, for instance, all the cash in the bond, the manager can always choose a strategy such that the constraint $W(T) - G_T \ge 0 \ P-a.e.$ is satisfied.

3 Main property of the surplus process

In this section, we define the surplus process. We prove that it is self-financing, and we deduce a market efficiency test for the pension fund.

The surplus process $Y(t)$, $t \ge 0$ is defined by:

$$Y(t) = W(t) + D(t) - G(t); \tag{13}$$

where

$$D(t) = E_t\left[\int_t^T \frac{H(s)}{H(t)} c_s ds\right]; \quad G(t) = E_t\left[\frac{H(T)}{H(t)} G_T\right]:$$

This process can be interpreted as a surplus process, in the sense that, at date t, it is equal to:

- the value of the portfolio $W(t)$
- plus the discounted value of the future engagements coming from the contributor $D(t)$,
- minus the discounted value of the pension fund future engagement (that is the guarantee) $G(t)$.

Note also that the value of the process at date T is equal to the surplus $W(T) - G_T$.

Proposition 2 The surplus process is self-financing, that is there exists a progressive measurable random process $(\underline{y}_t)_{t\in[0;T]} = \left(\left(1 - y_t^B - y_t^S\right); y_t^B; y_t^S\right)'$ such that:

$$dY_t = Y_t \underline{y}_t{}' \mathrm{diag}[\underline{S}_t]^{-1} d\underline{S}_t \tag{14}$$

Proof.

For a given process K_t let denote $\tilde{K}_t := H_t K_t$. Then :

$$d\tilde{Y}_t = d\tilde{W}_t + d\tilde{D}_t - d\tilde{G}_t$$

From (2), (5), and (11), easy computations lead to:

$$d\left(\tilde{W}_t\right) = \tilde{W}_t\left[\underline{b}_t'\sigma(t;r_t) - \underline{\lambda}'(t)\right] d\underline{z}_t + \tilde{e}_t dt$$

where $\underline{b}_t' = \left[u_t^B; u_t^S\right]$ and $\sigma(t;r_t) = \begin{pmatrix} 0 & \sigma_B(T-t;r_t)\sqrt{{}_1 r_t + {}_2} \\ \sigma_1 & \sigma_2 \end{pmatrix}$.

Using the martingale representation theorem for the Brownian motion, (see e.g. Karatzas and Shreve 1990), it turns out that there exists a unique square integrable process $(\underline{\theta}_t)_{t\in[0;T]}$ with $\underline{\theta}_t = (\theta_t; \theta_t^r)'$, satisfying

$$\int_0^T |\underline{\theta}_t|^2 dt < +\infty \quad P - a.e. \tag{15}$$

such that

$$d\left(\tilde{D}_t\right) = -\tilde{c}_t dt + \underline{\zeta}_t' d\underline{z}_t \tag{16}$$

Analogously, there exists a unique square integrable process $(\underline{\rho}_t)_{t\in[0,T]}$ with $\underline{\rho}_t = (\rho_t, \rho_t^r)'$, satisfying

$$\int_0^T |\underline{\rho}_t|^2 \, dt < +\infty \quad P - a.e. \tag{17}$$

such that

$$d\left(\tilde{G}_t\right) := d\left(E_t\left[\tilde{G}_T\right]\right) = \underline{\rho}_t' d\underline{z}(t)$$

Finally, we get:

$$d\tilde{Y}(t) = \left(\tilde{W}(t)\left[\underline{b}_t' \sigma(t, r_t) - \underline{\lambda}'(t)\right] + \underline{\zeta}_t' - \underline{\rho}_t'\right) d\underline{z}(t)$$

and therefore the process $Y(t)$ is self-financing. Indeed, in order to prove (14), it suffices to define $\underline{y}_t' = \left[y_t^B, y_t^S\right]$ as follows:

$$Y(t)\underline{y}_t = W(t)\underline{b}_t + (D_t - G_t)\left[\sigma(t, r_t)'\right]^{-1}\underline{\lambda}(t) + H^{-1}(t)\left[\sigma(t, r_t)'\right]^{-1}\left(\underline{\zeta}_t - \underline{\rho}_t\right) \tag{18}$$

which ends the proof. ■

By using the methodology of Deelstra, Grasselli and Koehl (2000), it is easy to find the explicit form of $\underline{b}_t' = \left[y_t^B, y_t^S\right]$ which only depends on the parameters of the financial market and not on the contributions nor the guarantee. The influence of the contributions and the guarantee is reflected by $D_t, G_t, \underline{\zeta}_t$ and $\underline{\rho}_t$. These can be studied under some explicit hypotheses upon the contributions and the guarantee: we give some examples in the Appendix.

The following corollary provides an exponential expression for the final surplus Y_T, that will be used extensively in the rest of the paper.

Corollary 3 The final surplus Y_T satisfies the following equation:

$$\begin{aligned} Y_T = \; & Y_0 \exp\left(\int_0^T r_s\left[1 - y_s^B - y_s^S\right] ds + \int_0^T y_s^B \frac{dB(s,T)}{B(s,T)} + \int_0^T y_s^S \frac{dS(s)}{S(s)} \right. \\ & \left. - \frac{1}{2}\int_0^T \left(\left[y_s^B \sigma_B(T - s, r_s)\right]^2 + \left[y_s^S \sigma_1\right]^2 + \left[y_s^S \sigma_2\right]^2 (\eta_1 r_s + \eta_2)\right) ds \right) \end{aligned} \tag{19}$$

with

$$Y_0 = W_0 + E\left[\int_0^T H(s)c_s ds\right] - E[H(T)G_T] \geq 0.$$

Note that Y_T depends on G_T through Y_0 only. From now on we will stress this dependence by denoting Y_T as a function of G_T.

Defining $\varphi\left(T; \underline{y}; \underline{S}\right)$ as follows

$$\begin{aligned}\varphi\left(T; \underline{y}; \underline{S}\right) = \exp\Bigg(& \int_0^T r_s\left(1 - y_s^B - y_s^S\right) ds + \int_0^T y_s^B \frac{dB(s;T)}{B(s;T)} + \int_0^T y_s^S \frac{dS(s)}{S(s)} \\ & - \frac{1}{2}\int_0^T \left[\left(y_s^B \sigma_B(T-s; r_s)\right)^2 + \left(y_s^S \sigma_1\right)^2 + \left(y_s^S \sigma_2\right)^2 (\eta_1 r_s + \eta_2)\right] ds\Bigg);\end{aligned}$$

(19) can be rewritten as

$$Y_T(G_T) = \varphi\left(T; \underline{y}; \underline{S}\right)\left(W_0' - E[H(T)G_T]\right). \qquad (20)$$

For two different minimum guarantees G_T^1 and G_T^2, we can write

$$Y_T(G_T^2) = Y_T(G_T^1)\frac{W_0' - E[H(T)G_T^2]}{W_0' - E[H(T)G_T^1]}. \qquad (21)$$

If we compare the surplus in case with a guarantee G_T with the no-guarantee-case, we have a strong relationship between the variances of the surplus and the expectation.

Proposition 4 (Market efficiency test) *There exists a constant k such that*

$$Var[Y_T(G_T)] = k\left(E[Y_T(G_T)]\right)^2.$$

Proof. From (21) it follows that

$$Var[Y_T(G_T)] = Var[Y_T(0)]\left(1 - \frac{E[H(T)G_T]}{W_0'}\right)^2 \qquad (22)$$

and

$$E[Y_T(G_T)] = E[Y_T(0)]\left(1 - \frac{E[H(T)G_T]}{W_0'}\right);$$

and therefore

$$\frac{Var[Y_T(G_T)]}{Var[Y_T(0)]} = \left(\frac{E[Y_T(G_T)]}{E[Y_T(0)]}\right)^2.$$

■

The last proposition delivers a strong test to check with data from some pension fund whether their investment strategies follow the model proposed in this paper.

4 The optimal guarantee for the contributor

By the analysis of the pension fund manager problem, we have obtained the principal features of the final surplus. Now, we come back to the initial problem of the contributor (8), that is:

$$\min_{G_T} Var\left[(1-\beta)Y_T(G_T)+G_T\right]$$

under the constraint

$$E\left[H(T)\left((1-\beta)Y_T(G_T)+G_T\right)\right]=k$$

Proposition 5 The solution of the contributor problem takes the following form:

$$G_T^{*}=\frac{\bar{k}}{E[H(T)]}+\frac{1-\beta}{2}\frac{cov(Y_T;H(T))}{E[H(T)]}+\frac{1-\beta}{2}\left(E[Y_T]-Y_T\right). \qquad (23)$$

with

$$\bar{k}=\frac{k-E\left[H(T)(1-\beta)\varphi\left(T;\underline{y};\underline{S}\right)W_0^0\right]}{1-E\left[H(T)(1-\beta)\varphi\left(T;\underline{y};\underline{S}\right)\right]}.$$

Proof. : Substituting the expression (20) of $Y_T(G_T)$ into the constraint of the optimization program, we find

$$E\left[H(T)\left((1-\beta)\varphi\left(T;\underline{y};\underline{S}\right)\left(W_0^0-E[H(T)G_T]\right)+G_T\right)\right]=k;$$

which is equivalent to the constraint

$$E[H(T)G_T]=\bar{k}.$$

Noticing that under this constraint,

$$Y_T(G_T)=\varphi\left(T;\underline{y};\underline{S}\right)\left(W_0^0-E[H(T)G_T]\right)=\varphi\left(T;\underline{y};\underline{S}\right)\left(W_0^0-\bar{k}\right);$$

one sees that the $Y_T(G_T)$ is a constant in this minimization problem with respect to G_T. Therefore, it will be denoted in this section Y_T only. Therefore, the problem is equivalent to

$$\min_{G_T}(1-\beta)cov(Y_T;G_T)+Var[G_T]$$

under the constraint $E[H(T)G_T] = \bar{k}$ or by using a Lagrangian multiplier

$$\min_{G_T} (1-\beta)\mathrm{cov}(Y_T, G_T) + Var[G_T] - \lambda\left(E[H(T)G_T] - \bar{k}\right).$$

This problem is solved by using variational calculus. Indeed, suppose that the minimum is attained at G_T^*. Then at $G_T^\varepsilon = G_T^* + \varepsilon\delta$ with $\delta \in L_+^p$ with $p > 2$ and $\varepsilon > 0$, one has

$$\frac{\partial}{\partial\varepsilon}\left[(1-\beta)\mathrm{cov}(Y_T, G_T^\varepsilon) + Var[G_T^\varepsilon] - \lambda\left(E[H(T)G_T^\varepsilon] - \bar{k}\right)\right]_{|\varepsilon=0} = 0.$$

Rewriting the variance and covariance, this is equivalent to

$$\frac{\partial}{\partial\varepsilon}\left[\begin{array}{l}(1-\beta)\left(E[Y_T G_T^\varepsilon] - E[Y_T]E[G_T^\varepsilon]\right) + E\left[(G_T^\varepsilon)^2\right] \\ -E[G_T^\varepsilon]^2 - \lambda\left(E[H(T)G_T^\varepsilon] - \bar{k}\right)\end{array}\right]_{|\varepsilon=0} = 0.$$

Using, the uniform integrability which is a consequence of the L^p integrability with $p > 2$, one finds

$$E\left[(1-\beta)(Y_T\delta - E[Y_T]\delta) + 2G_T^\varepsilon\delta - 2E[G_T^\varepsilon]\delta - \lambda H(T)\delta\right]_{|\varepsilon=0} = 0,$$

or equivalently, for all $\delta \in L_+^p$ with $p > 2$

$$E\left[\delta\left((1-\beta)(Y_T - E[Y_T]) + 2G_T^* - 2E[G_T^*] - \lambda H(T)\right)\right] = 0.$$

Therefore almost surely

$$G_T^* = E[G_T^*] + \frac{1-\beta}{2}(E[Y_T] - Y_T) - H_T\lambda. \qquad (24)$$

Taking expectations of both the right and left hand side leads to $\lambda = 0$ and

$$G_T^* = E[G_T^*] + \frac{1-\beta}{2}(E[Y_T] - Y_T). \qquad (25)$$

We conclude that since $\lambda = 0$, G_T^* will always take the form given in (25).

At last, we want to express $E[G_T^*]$ in function of $\bar{k}$. Noticing that

$$\begin{aligned}\bar{k} &= E[H(T)G_T^*] \\ &= E[G_T^*]E[H(T)] + \frac{1-\beta}{2}\left(E[Y_T]E[H(T)] - E[H_T Y_T]\right) \\ &= E[G_T^*]E[H(T)] + \frac{\beta-1}{2}\mathrm{cov}(Y_T, H(T))\end{aligned}$$

proves that

$$G_T^* = \frac{\bar{k}}{E[H(T)]} + \frac{1-\beta}{2}\frac{cov(Y_T; H(T))}{E[H(T)]} + \frac{1-\beta}{2}(E[Y_T] - Y_T);$$

which ends the proof. ■

Corollary 6 Given the stochastic guarantee solution of the optimization program (6), the benefit is equal to

$$\frac{1-\beta}{2} Y_T + A\left(T; \underline{y}; \underline{S}; W_0^0\right)$$

with

$$A\left(T; \underline{y}; \underline{S}; W_0^0\right) = \frac{1}{B(0;T)}\left(\frac{1-\beta}{2} W_0^0 + \frac{1+\beta}{2}\bar{k}\right)$$

$$\bar{k} = \frac{k - E\left[H(T)(1-\beta)\varphi\left(T; \underline{y}; \underline{S}\right) W_0^0\right]}{1 - E\left[H(T)(1-\beta)\varphi\left(T; \underline{y}; \underline{S}\right)\right]}:$$

Proof : This result follows immediately from (23) and since $E[H(T)] = B(0;T)$ and $E[Y_T H(T)] = W_0^0 - \bar{k}$. ■

This last result proves that the optimal benefit is composed of two parts: a constant part and a fixed fraction of the surplus. From the contributor point of view, this is equivalent to obtain a constant guarantee $A\left(T; \underline{y}; \underline{S}; W_0^0\right)$ and a fraction $1 - \beta^0$ of the surplus, with $\beta^0 = \frac{1+\beta}{2}$: Thus, the surplus part of the benefit is the only random part.

5 Conclusion

In a Defined Contribution framework, we obtained the optimal guarantee that minimizes the variance of the benefit for a fixed market value. The key result of this paper is the self-financing feature of the surplus process which is proved under very general hypotheses upon the interest rates dynamics and for a financial market involving bonds and equity.

Surprisingly, we found that the benefit can be decomposed into two parts: one part that is constant and the other one that is a fixed fraction of the surplus. Given the fraction β of the surplus that will be retained by the pension fund manager and the level of contributions, it suffices from the contributor point of view, to compare the …xed part of the optimal bene…t in order to evaluate the best contract among those proposed in a concurrential pension fund market .

Moreover, under some hypotheses on the contribution process, it is possible to analyze numerically the dependence of the …xed part of the optimal bene…t upon di¤erent parameters of our model.

A natural extension of this work, would be to allow for the case of incomplete markets, since the contribution process, which is usually linked to the salary, is not necessarily generated by the market.

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Appendix: the optimal strategy for the pension fund in the CIR framework

The aim in this Appendix is to show a methodology for finding explicitly the solution of the pension fund problem $(\underline{\mathbf{b}}_t)_{t\in[0;T)} = \left(u_t^B; u_t^S\right)_{t\in[0;T)}$ given by (18), when the contribution process c_t and the guarantee G_T assume some interesting stochastic features. This will permit the fund manager to quantify the impacts on his investment strategy due to variations of the contribution policy and guarantee.

In order to avoid further technicalities, let us consider the Cox-Ingersoll-Ross dynamics

$$dr_t = (a - br_t)dt - \sigma_r \sqrt{r_t}dz_r(t); \quad t \geq 0; \tag{26}$$

with market price of risk given by $\lambda' = \left(\lambda_1; \lambda_2\sqrt{r_t}\right)$. In this case we can apply directly the result of Deelstra et al. (2000), who under the technical assumption that γ satisfies

$$\frac{\gamma}{1-\gamma}\left(1 + \frac{\lambda_2^2}{2} - \frac{\lambda_2 b}{\sigma_r}\right) \leq \frac{b^2}{2\sigma_r^2};$$

found that $(\underline{y}_t)_{t\in[0;T)} = \left(y_t^B; y_t^S\right)_{t\in[0;T)}$ is given explicitly by

$$y_t^B = \frac{k_2\left(T-t; \frac{\gamma}{1-\gamma}\right)}{h(T-t)} + \frac{1}{1-\gamma}\frac{\sigma_1\lambda_2 - \sigma_2\lambda_1}{\sigma_1\sigma_r h(T-t)};$$

$$y_t^S = \frac{1}{1-\gamma}\frac{\lambda_1}{\sigma_1};$$

with

$$k_2(t;c) = -c\frac{(e^{\alpha t}-1)\left(2 + \lambda_2^2(1+c)\right)}{\alpha - b - c\lambda_2\sigma_r + e^{\alpha t}(\alpha + b + c\lambda_2\sigma_r)};$$

$$\alpha = \sqrt{b^2 + 2\sigma_r^2\mu};$$

$$\mu = -c\left(1 + \frac{\lambda_2^2}{2} - \frac{\lambda_2 b}{\sigma_r}\right):$$

So, in order to find $(\underline{u}_t)_{t\in[0;T)} = \left(u_t^B; u_t^S\right)_{t\in[0;T)}$, it suffices to find the processes $\left(D_t; (\underline{\zeta}_t)\right)_{t\in[0;T]}$ and $\left(G_t; (\underline{\rho}_t)\right)_{t\in[0;T]}$.

Let us consider the following quite general (stochastic) contribution process:

$$c_t = c_0 \exp\left\{\int_0^t (\alpha_1(s) + \alpha_2 r_s)\, ds + \int_0^t \alpha_3 dz(s) + \int_0^t \alpha_4\sqrt{r_s}dz_r(s)\right\}; \tag{27}$$

with $\alpha_1(:)$ being a deterministic function and $\alpha_2; \alpha_3; \alpha_4$ being real constants. That is, we make the hypothesis that the contributions at time t can depend on the entire past salary history.

Moreover, let us consider an interest rate guarantee (see e.g. Jensen and Sørensen 2000): the pension fund assures a deterministic positive interest rate $(g_t)_{t\in[0,T]}$, so that the guarantee G_T becomes

$$G_T = W_0 \exp\left\{\int_0^T g_t dt\right\} + \int_0^T c_t \exp\left\{\int_t^T g_s ds\right\} dt. \qquad (28)$$

Obviously, there must be some admissibility constraint on $(g_t)_{t\in[0,T]}$, in order to avoid arbitrage opportunities: for notational reasons we will show this condition in the end of the Appendix.

Let us firstly consider the problem of finding $\left(D_t, (\zeta_t)_{t\in[0,T]}\right)$.

Proposition 7 Suppose that the contribution process is given by (27) and that the following relation holds:

$$1 + \frac{1}{2}\lambda_2^2 - \alpha_2 + \frac{\alpha_4 - \lambda_2}{\sigma_r} b + \frac{b^2}{2\sigma_r^2} \geq 0.$$

Then

$$\begin{aligned} D_t &= \frac{1}{H(t)} E_t\left[\int_t^T H(s) c_s ds\right] \\ &= c_t \int_t^T \exp\left\{-A_1(t,s) - A_2(t,s) r_t\right\} ds \end{aligned} \qquad (29)$$

with

$$A_1(t,s) = a\varphi_{\eta_c,\mu_c}(s-t) - \int_t^s \left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 + \frac{\alpha_4 - \lambda_2}{\sigma_r} a + \frac{(\alpha_3 - \lambda_1)^2}{2}\right) du; \qquad (30)$$

$$A_2(t,s) = \psi_{\eta_c,\mu_c}(s-t) - \frac{\alpha_4 - \lambda_2}{\sigma_r}; \qquad (31)$$

$$\varphi_{\eta_c,\mu_c}(t) = -\frac{2}{\sigma_r^2}\log\left(\frac{2\xi e^{\frac{(\xi+b)t}{2}}}{\sigma_r^2\eta_c(e^{\xi t}-1) + \xi - b + e^{\xi t}(\xi+b)}\right);$$

$$\varphi'_{\eta_c,\mu_c}(t) = \psi_{\eta_c,\mu_c}(t) = \frac{\eta_c\left[\xi + b + e^{\xi t}(\xi - b)\right] + 2\mu_c\left[e^{\xi t} - 1\right]}{\sigma_r^2\eta_c(e^{\xi t}-1) + \xi - b + e^{\xi t}(\xi+b)}; \qquad (32)$$

$$\mu_c = 1 + \frac{1}{2}\lambda_2^2 + \frac{\alpha_4 - \lambda_2}{\sigma_r} b - \alpha_2; \qquad (33)$$

$$\eta_c = \frac{\alpha_4 - \lambda_2}{\sigma_r}; \qquad (34)$$

and $\xi = \sqrt{b^2 + 2\sigma_r^2 1 c}$.

Proof. From (5) it results

$$H(t) = \exp\left\{ -\int_0^t \left(r_s + \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 r_s\right) \right) ds - \int_0^t \lambda_1 dz(s) - \int_0^t \lambda_2 \sqrt{r_s}\, dz_r(s) \right\};$$

so that

$$\begin{aligned} D_t &= c_t \int_t^T E_t\left[\exp\left\{ \int_t^s \left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 + r_u\left(\alpha_2 - 1 - \frac{\lambda_2^2}{2}\right)\right) du + \right.\right. \\ &\quad \left.\left. + \int_t^s (\alpha_3 - \lambda_1) dz(u) + \int_t^s (\alpha_4 - \lambda_2)\sqrt{r_u}\, dz_r(u) \right\} \right] ds. \end{aligned}$$

By independence of $z(t)$ and $(r_t; z_r(t))$,

$$\begin{aligned} D_t &= c_t \int_t^T \left\{ \exp\left\{ \int_t^s \left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 \right) du \right\} E_t\left[\exp\left\{ \int_t^s (\alpha_3 - \lambda_1) dz(u) \right\} \right] \right. \\ &\quad \left. \cdot E_t\left[\exp\left\{ \int_t^s \left(\alpha_2 - 1 - \frac{1}{2}\lambda_2^2 \right) r_u du + \int_t^s (\alpha_4 - \lambda_2)\sqrt{r_u}\, dz_r(u) \right\} \right] \right\} ds. \end{aligned}$$

Substituting the dynamics (26) of the interest rates, we obtain

$$\int_t^s (\alpha_4 - \lambda_2)\sqrt{r_u}\, dz_r(u) = \frac{\alpha_4 - \lambda_2}{\sigma_r} \int_t^s (a - b r_u) du - \frac{\alpha_4 - \lambda_2}{\sigma_r}(r_s - r_t);$$

and

$$\begin{aligned} D_t &= c_t \int_t^T \left\{ \exp\left\{ \int_t^s \left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 + \frac{\alpha_4 - \lambda_2}{\sigma_r} a \right) du + \frac{\alpha_4 - \lambda_2}{\sigma_r} r_t \right\} \right. \\ &\quad \cdot \exp\left\{ \frac{(\alpha_3 - \lambda_1)^2}{2}(s - t) \right\} \\ &\quad \left. \cdot E_t\left[\exp\left\{ -\frac{\alpha_4 - \lambda_2}{\sigma_r} r_s - \left(1 + \frac{\lambda_2^2}{2} + b\frac{\alpha_4 - \lambda_2}{\sigma_r} - \alpha_2 \right) \int_t^s r_u du \right\} \right] \right\} ds \\ &= c_t \exp\left\{ \frac{\alpha_4 - \lambda_2}{\sigma_r} r_t \right\} \\ &\quad \cdot \int_t^T \left\{ \exp\left\{ \int_t^s \left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 + \frac{\alpha_4 - \lambda_2}{\sigma_r} a + \frac{(\alpha_3 - \lambda_1)^2}{2} \right) du \right\} \right. \\ &\quad \left. \cdot E_t\left[\exp\left\{ -\frac{\alpha_4 - \lambda_2}{\sigma_r} r_s - \left(1 + \frac{\lambda_2^2}{2} + b\frac{\alpha_4 - \lambda_2}{\sigma_r} - \alpha_2 \right) \int_t^s r_u du \right\} \right] \right\} ds. \end{aligned}$$

We apply now the formula of Pitman and Yor (1982) (for a discussion about the coefficients, see Deelstra et al. 2000), which gives us the Laplacian transform of the couple $\left(r_s, \int_t^s r_u du\right)$:

$$E_t\left[\exp\left\{-\eta r_T - \mu\int_t^T r_s ds\right\}\right] = \exp\left\{-a\phi_{\eta,\mu}(T-t) - \psi_{\eta,\mu}(T-t)r_t\right\}.$$

We replace (μ, η) with (μ^c, η^c) and obtain (29). ■

Proposition 8 Under the hypotheses of the previous proposition, the process $(\underline{\zeta}_t)_{t\in[0,T]} = (\zeta_t, \zeta_t^r)'$ is given by

$$\begin{aligned}
\zeta_t &= H(t)D_t(\alpha_3 - \lambda_1);\\
\zeta_t^r &= \sqrt{r_t}H(t)\left\{D_t(\alpha_4 - \lambda_2) + c_t\sigma_r\left(\int_t^T A_2(t,s)\exp\{-A_1(t,s) - A_2(t,s)r_t\}ds\right)\right\}.
\end{aligned}$$

Proof. From (16) it turns out that ζ_t (resp. ζ_t^r) is the coefficient of $dz(t)$ (resp. $dz_r(t)$) in the development of $d\tilde{D}_t = d(H(t)D_t)$, so that we can group the (locally) deterministic factors into $[.]dt$ and focus on the others:

$$\begin{aligned}
d\tilde{D}_t &= [.]dt + H(t)dD_t + D_t dH(t)\\
&= [.]dt + H(t)dD_t - H(t)D_t\underline{\eta}'(t,r_t)d\underline{z}(t)
\end{aligned}$$

Now we have

$$dD_t = [.]dt + D_t\frac{dc_t}{c_t} + c_t\sigma_r\left(\int_t^T A_2(t,s)\exp\{-A_1(t,s) - A_2(t,s)r_t\}ds\right)\sqrt{r_t}dz_r(t);$$

so from (27), we obtain

$$\begin{aligned}
dD_t &= [.]dt + D_t\alpha_3 dz(t)\\
&\quad + \left[D_t\alpha_4 + c_t\sigma_r\left(\int_t^T A_2(t,s)\exp\{-A_1(t,s) - A_2(t,s)r_t\}ds\right)\right]\sqrt{r_t}dz_r(t).
\end{aligned}$$

Finally,

$$\begin{aligned}
d\tilde{D}_t &= [.]dt + H(t)D_t(\alpha_3 - \lambda_1)dz(t) + H(t)D_t(\alpha_4 - \lambda_2)\sqrt{r_t}dz_r(t)\\
&\quad + H(t)c_t\sigma_r\left(\int_t^T A_2(t,s)\exp\{-A_1(t,s) - A_2(t,s)r_t\}ds\right)\sqrt{r_t}dz_r(t)\\
&= -H(t)c_t dt + H(t)D_t(\alpha_3 - \lambda_1)dz(t) + H(t)D_t(\alpha_4 - \lambda_2)\sqrt{r_t}dz_r(t)\\
&\quad + H(t)c_t\sigma_r\left(\int_t^T A_2(t,s)\exp\{-A_1(t,s) - A_2(t,s)r_t\}ds\right)\sqrt{r_t}dz_r(t)
\end{aligned}$$

where the last equality follows from (16). ■

By the same methodology, we can determine the processes $\left(G_t; (\underline{\rho}_t)\right)_{t\in[0;T]}$ for the guarantee G_T given by (28):

$$
\begin{aligned}
G_t &= E_t\left[\frac{H(T)}{H(t)}G_T\right] \\
&= \frac{1}{H(t)}E_t\left[H(T)W_0\exp\left\{\int_0^T g_t dt\right\} + \int_0^T H(T)c_t\exp\left\{\int_t^T g_s ds\right\}dt\right] \\
&= \left[W_0\exp\left\{\int_0^T g_t dt\right\} + \int_0^t c_s\exp\left\{\int_s^T g_u du\right\}ds\right]B(t;T) \\
&\quad + \int_t^T \exp\left\{\int_s^T g_u du\right\}E_t\left[\frac{H(T)}{H(t)}c_s\right]ds.
\end{aligned}
$$

Now,

$$
\begin{aligned}
E_t\left[H(T)c_s\right] &= E_t\left[H(s)c_s E_s\left[\frac{H(T)}{H(s)}\right]\right] \\
&= E_t\left[H(s)c_s B(s;T)\right];
\end{aligned}
$$

so from (3), we obtain

$$
\begin{aligned}
E_t\left[\frac{H(T)}{H(t)}c_s\right] &= c_t E_t \exp\left\{-\int_t^s\left(r_u + \frac{1}{2}\|\underline{\lambda}(u)\|^2\right)du - \int_t^s \underline{\lambda}'(u)d\underline{z}(u)\right. \\
&\quad + \int_t^s(\alpha_1(u)+\alpha_2 r_u)\,du + \int_t^s \alpha_3 dz(u) + \int_t^s \alpha_4\sqrt{r_u}dz_r(u) \\
&\quad \left. - a\phi(T-s) - r_s h(T-s)\right\} \\
&= c_t\exp\left(\int_t^s\left(\alpha_1(u) - \frac{1}{2}\lambda_1^2\right)du - a\phi(T-s) + \frac{(\alpha_3-\lambda_1)^2}{2}(s-t)\right) \\
&\quad .\exp\left\{\frac{a}{\sigma_r}(\alpha_4-\lambda_2)(s-t) + \frac{\alpha_4-\lambda_2}{\sigma_r}r_t\right\} \\
&\quad .E_t\exp\left\{-\hat{\eta}^c r_s - 1^c\int_t^s r_u du\right\};
\end{aligned}
$$

with

$$
\begin{aligned}
\hat{\eta}^c &= \eta^c + h(T-s) \\
&= \frac{\alpha_4-\lambda_2}{\sigma_r} + h(T-s);
\end{aligned}
$$

and μ^c is given by (33). We apply Pitman and Yor (1982) and obtain

$$E_t\left[\frac{H(T)}{H(t)}c_s\right] = c_t \exp\left\{-\hat{A}_1(t;s;T) - \hat{A}_2(t;s)r_t\right\};$$

where the deterministic functions $\hat{A}_1(t;s;T)$ and $\hat{A}_2(t;s)$ are defined by analogy with (30) and (31) with η^c replaced by $\hat{\eta}^c$:

$$\hat{A}_1(t;s;T) = a\phi_{\hat{\eta}^c,\mu^c}(s-t) - \int_t^s \left(\alpha_1(u) - \frac{1}{2}\lambda_1^2 + \frac{\alpha_4-\lambda_2}{\sigma_r}a + \frac{(\alpha_3-\lambda_1)^2}{2}\right)du;$$

$$\hat{A}_2(t;s) = \psi_{\hat{\eta}^c,\mu^c}(s-t) - \frac{\alpha_4-\lambda_2}{\sigma_r}.$$

Finally, we have

$$\begin{aligned} G_t = {} & \left[W_0 \exp\left\{\int_0^T g_t dt\right\} + \int_0^t c_s \exp\left\{\int_s^T g_u du\right\} ds\right] B(t;T) \\ & + c_t \int_t^T \exp\left\{\int_s^T g_u du - \hat{A}_1(t;s;T) - \hat{A}_2(t;s)r_t\right\} ds; \end{aligned} \qquad (35)$$

and

$$d(H(t)G_t) = [\cdot]dt + H(t)dG_t - H(t)G_t\underline{\lambda}'(t)d\underline{z}(t).$$

Since

$$\begin{aligned} dG_t = {} & [\cdot]dt + \left[W_0 \exp\left\{\int_0^T g_t dt\right\} + \int_0^t c_s \exp\left\{\int_s^T g_u du\right\} ds\right] B(t;T)\sigma_r h(T-t)\sqrt{r_t}dz_r(t) \\ & + c_t \int_t^T \exp\left\{\int_s^T g_u du - \hat{A}_1(t;s;T) - \hat{A}_2(t;s)r_t\right\} ds\,(\alpha_3 dz(t) + \alpha_4\sqrt{r_t}dz_r(t)) \\ & + c_t\sigma_r \left(\int_t^T \hat{A}_2(t;s)\exp\left\{\int_s^T g_u du - \hat{A}_1(t;s;T) - \hat{A}_2(t;s)r_t\right\} ds\right)\sqrt{r_t}dz_r(t); \end{aligned}$$

then the process $(\underline{\rho}_t)_{t\in[0;T]}$ follows:

$$\rho_t = H(t)c_t\alpha_3 \int_t^T \exp\left\{\int_s^T g_u du - \hat{A}_1(t;s;T) - \hat{A}_2(t;s)r_t\right\} ds - H(t)G_t\lambda_1;$$

$$\begin{aligned} \rho_t^r = {} & \sqrt{r_t}H(t)\left\{\left[B(t;T)\left[W_0 \exp\left\{\int_0^T g_t dt\right\} + \int_0^t c_s \exp\left\{\int_s^T g_u du\right\} ds\right]\sigma_r h(T-t)\right.\right. \\ & + c_t\alpha_4 \left(\int_t^T \exp\left\{\int_s^T g_u du - \hat{A}_1(t;s;T) - \hat{A}_2(t;s)r_t\right\} ds\right) \\ & \left.\left. + c_t\sigma_r \left(\int_t^T \hat{A}_2(t;s)\exp\left\{\int_s^T g_u du - \hat{A}_1(t;s;T) - \hat{A}_2(t;s)r_t\right\} ds\right)\right] - \lambda_2 G_t\right\}. \end{aligned}$$

For the particular case of deterministic contributions (i.e. with $\alpha_2 = \alpha_3 = \alpha_4 = 0$), it turns out that:

$$dD_t = -c_t dt + \int_t^T c_s (dB(t;s))\, ds$$

and that the process $(\underline{\zeta}_t)_{t\in[0;T]} = (\zeta_t; \zeta_t^r)'$ is given by

$$\begin{aligned}
\zeta_t &= -H(t) D_{t,1};\\
\zeta_t^r &= \int_t^T c_s B(t;s)\sigma_B(s-t;r_t)ds - H(t)D_{t}\lambda_2\sqrt{r_t};
\end{aligned}$$

while, when the guarantee G_T is a strictly positive constant, it is easy to check that the process $(\underline{\rho}_t)_{t\in[0;T]} = (\rho_t; \rho_t^r)'$ is given by

$$\begin{aligned}
\rho_t &= -H(t)G_T B(t;T)\lambda_1;\\
\rho_t^r &= (\sigma_B(T-t;r_t) - \lambda_2\sqrt{r_t})\, H(t)G_T B(t;T):
\end{aligned}$$

As mentioned above, we end the Appendix by showing the admissibility condition on the interest rate g_t: from (12) it follows

$$W_0 e^{\int_0^T g_t dt} B(0;T) + \int_0^T e^{\int_t^T g_s ds} E\left[\frac{c_t}{H(t)} B(t;T)\right] dt < W_0 + D_0;$$

then

$$W_0 e^{\int_0^T g_t dt} B(0;T) + c_0 \int_0^T e^{\int_t^T g_s ds - \tilde{A}_1(0;t;T) - \tilde{A}_2(0;t) r_0} dt < W_0 + D_0;$$

which defines an upper bound for the possible values of the deterministic process $(g_t)_{t\in[0;T]}$.