AMERICAN OPTIONS WITH REGIME SWITCHING

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Abstract. A Black-Scholes market is considered in which the underlying economy, as modelled by the parameters and volatility of the processes, switches between a finite number of states. The switching is modelled by a hidden Markov chain. European options are priced and a Black-Scholes equation obtained. The approximate valuation of American options due to Barone-Adesi and Whaley is extended to this setting.

Key Words: Option pricing, free boundary problem, Black-Scholes equation.

Acknowledgement: This work was partially supported by the Social Sciences and Humanities Research Council of Canada.

1. Introduction

We consider the standard Black-Scholes market with a bond and risky asset. However, we suppose the bank interest rate, and appreciation rate and volatility of the risky asset, depend on the state of the economy. This is modeled by a finite state Markov chain, X.

European options have been discussed in this framework by Di Masi et al. [2] and Guo [5]. Russian options and perpetual American options have also been treated by Guo.

Our treatment of the European option and derivation of the characteristic function of occupation times is different. We also give an extension to the regime switching framework of the Barone-Adesi-Whaley, [1], approximation for the value of American options. Details of the calculations are given in the three Appendices.

We suppose the economic state of the world is described by a finite state Markov chain $X = \{X_t, t \geq 0\}$. In particular, there could be just two states for X representing 'good' and 'bad'.

As in [3], the state space of X can be taken to be, without loss of generality, the set of unit vectors $\{e_1, e_2, \ldots, e_N\}$, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)' \in R^N$.

Suppose the process X is homogeneous in time and has a rate matrix A. Then if $p_t = E[X_t] \in R^N$

$$\frac{dp_t}{dt} = Ap_t.$$

As in [3] we can then show that

$$X_t = X_0 + \int_0^t AX_u du + M_t \tag{1}$$

where $M = \{M_t,\ t \geq 0\}$ is a martingale with respect to the filtration generated by X.

Our financial market has two underlying assets, a bank account and a risky asset.

We suppose the instantaneous interest rate in the bank depends on the state X of the economy, so that

$$r_t = \langle r, X_t \rangle$$

where $r = (r_1, r_2, \ldots, r_N)' \in R^N$.

Then \$1 invested at time zero in the bank becomes

$$S_t^0 = \exp\Big(\int_0^t r_u du\Big) \tag{2}$$

at time t.

Similarly, suppose the rate of return $\mu = \{\mu_t,\ t \geq 0\}$ and volatility $\sigma = \{\sigma_t,\ t \geq 0\}$ depend on the state $X = \{X_t,\ t \geq 0\}$ of the economy. That is, there are vectors $\mu = (\mu_1, \ldots, \mu_N)'$ and $\sigma = (\sigma_1, \ldots, \sigma_N)' \in R^N$ such that

$$\mu_t = \langle \mu, X_t \rangle$$

$$\sigma_t = \langle \sigma, X_t \rangle.$$

We suppose $\sigma_i > 0, \quad i = 1, \ldots, N$. The price of the risky asset $S^1 = \{S_t^1,\ t \geq 0\}$ is then given by the dynamics

$$S_t^1 = S_0^1 + \int_0^t \mu_u S_u^1 du + \int_0^t \sigma_u S_u^1 dB_u. \tag{3}$$

Here $B = \{B_t,\ t \geq 0\}$ is a Brownian motion on $(\Omega, \mathcal{F}, P)$. Write $\{\mathcal{F}_t\}$ for the filtration generated by S.

The solution of (3) is

$$S_t^1 = S_0^1 \exp\Big(\int_0^t (\mu_u - \frac{\sigma_u^2}{2}) du + \int_0^t \sigma_u dB_u^u\Big).$$

The extra uncertainty introduced by the regime switching means our market is no longer complete. This question is discussed in Guo [6], where the market is completed by

Arrow-Debreu securities related to the cost of switching. We shall not repeat this argument but assume we are already working under a risk neutral measure P so that the price of a European claim $g(S^1_T)$ with exercise time T is

$$E\Big[\exp\Big(-\int_0^T r_u du\Big) g(S^1_T)|\mathcal{F}_0\Big].$$

In particular, the price of a European call at time $t \in [0, T]$ is

$$C(t, T, S^1_0, X_0) = E\Big[\exp\Big(-\int_t^T r_u du\Big)\big(S^1_T - K\big)^+|\mathcal{F}_t\Big] \tag{4}$$

where

$$S^1_T = S^1_t \exp\Big(\int_t^T (\mu_u - \frac{\sigma_u^2}{2})du + \int_t^T \sigma_u dB_u\Big). \tag{5}$$

If the processes r, μ, σ are deterministic and independent of X then

$$C(t, T, S_0) = S^1_t N(d_1) - e^{-\int_t^T r_u du} K N(d_2) \tag{6}$$

where

$$d_1(t) = \Big(\int_t^T \sigma_u^2 du\Big)^{-1/2}\Big[\log \frac{S^1}{K} + \Big(\int_t^T r_u du + \frac{1}{2}\int_t^T \sigma_u^2 du\Big)\Big]$$

and $\quad d_2(t) = d_1(t) - \Big(\int_t^T \sigma_u^2 du\Big)^{1/2}.$

Suppose now the processes r, μ, σ depend on X so that

$$r_t = \langle r, X_t\rangle, \quad \mu_t = \langle \mu, X_t\rangle, \quad \sigma_t = \langle \sigma, X_t\rangle\,.$$

If we knew the trajectory of X between time 0 and time T we would know the values of

$$P_{t,T} = \int_t^T \langle r, X_u\rangle du, \quad L_{t,T} = \int_t^T \langle \mu, X_u\rangle du, \quad U_{t,T} = \int_t^T \langle \sigma, X_u\rangle^2 du.$$

That is, if $\mathcal{F}_t = \sigma\{X_u,\ 0 \le u \le t\}$

$$C(t, T, S_t^1, P_T, U_T) = E\Big[\exp\big(-P_{t,T}\big)\big(S_T^1 - K\big)^+ \big|\mathcal{F}_T\Big] \tag{7}$$

$$= S_t^1 N\big(d_1(t)\big) - \exp\big(-P_{t,T}\big) K N\big(d_2(t)\big) \tag{8}$$

and

$$d_1(t) = (U_{t,T})^{-1/2}\Big[\log \frac{S_t^1}{K} + P_{t,T} + \frac{1}{2} U_{t,T}\Big]$$

$$d_2(t) = d_1(t) - U_{t,T}^{1/2}.$$

To determine the call option price in this model we must take a second expectation over the variables $P_{t,T}$ and $U_{t,T}$.

For now suppose $t = 0$. For $1 \le i \le N$ let

$$T_i = \int_0^T \langle e_i, X_u\rangle du$$

be the amount of time X has spent in state e_i up to time T. Then $T_1 + T_2 + \cdots + T_N = T$ so we can discuss just $T_1, T_2, \ldots, T_{N-1}$. Also

$$P_{0,T} = P_T = \int_0^T \langle r, X_u\rangle du$$

$$= \sum_{i=1}^{N} r_i T_i = \sum_{i=1}^{N-1} (r_i - r_N) T_i + r_N T \tag{9}$$

and

$$U_{0,T} = U_T = \int_0^T \langle \sigma, X_u\rangle^2 du = \sum_{i=1}^{N} \sigma_i^2 T_i$$

$$= \sum_{i=1}^{N-1} (\sigma_i^2 - \sigma_N^2) T_i + \sigma_N^2 T. \tag{10}$$

Suppose $\phi(\tau_1, \tau_2, \ldots, \tau_{N-1})$ is the density function of $(T_1, T_2, \ldots, T_{N-1})$. Then

$$C(0, T, S_0^1, X_0) = \int_0^T \cdots \int_0^T C(0, T, S_0^1, P_T, U_T)\phi(\tau_1, \tau_2, \ldots, \tau_{N-1})d\tau_1 \ldots d\tau_{N-1}.$$

We shall determine the Fourier transform of ϕ, that is, the characteristic function of $(T_1, T_2, \ldots, T_{N-1})$.

For any $\theta = (\theta_1, \theta_2, \ldots, \theta_{N-1}) \in R^{N-1}$ the characteristic function of $(T_1, T_2, \ldots, T_{N-1})$ is

$$E\Big[\exp i\Big(\sum_{j=1}^{N-1} \theta_j T_j\Big)\Big]_T = E\Big[\exp i\Big(\int_0^T \langle \boldsymbol{\theta}, X_u\rangle du\Big)\Big]$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{N-1}, 0) \in R^N$.

An explicit expression for the characteristic function is derived in Appendix 1 where it is shown that:

$$E\Big[\exp i\Big(\sum_{j=1}^{N-1} \theta_j T_j\Big)\Big] = \langle \exp[(A + i\,\mathrm{diag}\,\boldsymbol{\theta})T]X_0, \mathbf{1}\rangle\,.$$

2. The Black Scholes Equation

In this section we derive the system of partial differential equations satisfied by the option price. Suppose, as above, the state of the economy is determined by the finite state Markov chain $X = \{X_t,\, t \geq 0\}$, $X_t \in \{e_1, \ldots, e_N\}$ with dynamics

$$X_t = X_0 + \int_0^t AX_s ds + M_t. \tag{11}$$

With $r_t = \langle r, X_t\rangle$, $\mu_t = \langle \mu, X_t\rangle$,

$$\sigma_t = \langle \sigma, X_t\rangle$$

our market includes the two underlying assets S^0 and S^1 where

$$S_t^0 = \exp\Big(\int_0^t r_u du\Big) \tag{12}$$

$$S_t^1 = S_0^1 \exp\Big(\int_0^t (\mu_u - \frac{\sigma_u^2}{2}) du + \int_0^t \sigma_u dB_u\Big). \tag{13}$$

If at time $t \in [0, T]$, $S_t^1 = S$ and $X_t = X$ the price of a European call option with exercise date T and strike K is

$$C(t, T, S, X) = E\Big[\exp\Big(-\int_t^T r_u du\Big)\big(S_T^1 - K\big)^+ \big| S_t^1 = S,\, X_t = X\Big]. \tag{14}$$

This is evaluated similarly to $C(0, T, S, X)$.

Write

$$\begin{aligned} V(t, S, X) &= \exp\Big(-\int_0^t r_u du\Big) C(t, T, S, X) \\ &= E\Big[\exp\Big(-\int_0^T r_u du\Big)\big(S_T^1 - K\big)^+ \big| S_t^1 = S,\, X_t = X\Big] \\ &= E\Big[\exp\Big(-\int_0^T r_u du\Big)\big(S_T^1 - K\big)^+ \big| \mathcal{G}_t\Big] \end{aligned} \tag{15}$$

where $\mathcal{G}_t = \sigma\{S_u^1,\, X_u : u \leq t\}$. Consequently V is a $\mathcal{G}_t$-martingale. Write

$$\boldsymbol{V}(t, S) = \big(V(t, S, e_1), \ldots, V(t, S, e_2)\big),$$

so that $V(t, S_t^1, X_t) = \langle \boldsymbol{V}(t, S_t^1), X_t \rangle$. Applying the Itô rule to V we have

$$\begin{aligned} V(t, S_t^1, X_t) &= V(0, S_t^1, X_t) + \int_0^t \frac{\partial V}{\partial u} du \\ &\quad + \int_0^t \frac{\partial V}{\partial s} (\mu_u S_u^1 du + \sigma_u S_u^1 dB_u) \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 V}{\partial S^2} \sigma_u^2 (S_u^1) du + \int_0^t \langle \boldsymbol{V}, dX_t \rangle \end{aligned} \tag{16}$$

and $dX_t = AX_t dt + dM_t$.

By definition V is a martingale, so all the time integral terms in (16) must sum to zero identically. That is,

$$\frac{\partial V}{\partial t} + \mu_t S^1 \frac{\partial V}{\partial S} + \frac{1}{2}(\sigma_t S^1)^2 \frac{\partial^2 V}{\partial S^2} + \langle \boldsymbol{V}, AX \rangle = 0. \tag{17}$$

Now $V = \exp\big(-\int_0^t r_u du\big)C$, so with

$$\boldsymbol{C}(t,S) = \big(C(t,T,S,e_1), \ldots, C(t,T,S,e_N)\big)$$

we have

$$\Big(\exp\big(-\int_0^t r_u du\big)\Big)\Big(-r_t C + \frac{\partial C}{\partial t} + \mu_t S^1 \frac{\partial C}{\partial S} + \frac{1}{2}(\sigma_t S^1)^2 \frac{\partial^2 C}{\partial S^2} + \langle \boldsymbol{C}, AX \rangle\Big) = 0 \tag{18}$$

with terminal condition

$$C(T,T,S,X) = (S-K)^+.$$

Equation (18) reduces to N equations with $X = e_1, e_2, \ldots, e_N$. That is, with $X = X_t = e_i$

$$r_t = \langle r, X_t \rangle = r_i$$

$$\mu_t = \langle \mu, X_t \rangle = \mu_i$$

$$\text{and} \quad \sigma_t = \langle \sigma, X_t \rangle = \sigma_i,$$

and, writing

$$C_i = C(t,T,S,e_i), \quad \boldsymbol{C} = (C_1, C_2, \ldots, C_N)$$

we have that $\boldsymbol{C}$ satisfies the system of coupled Black-Scholes equations

$$-r_i C_i + \frac{\partial C_i}{\partial t} + \mu_i S \frac{\partial C_i}{\partial S} + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 C_i}{\partial S^2} + \langle \boldsymbol{C}, A e_i \rangle = 0 \tag{19}$$

with terminal condition

$$C(T,T,S,e_i) = (S-K)^+.$$

3. American Options with Switching

Consider now the American put in this regime switching model.

We shall simplify our discussion by supposing $N = 2$.

We suppose the expiration time is T and we suppose the put can be exercised at any stopping time $\tau \le T$. The payoff is then $(K - S^1_\tau)^+$.

The value function for the American put is

$$D(t,T,S,X) = \sup_{t\le\tau\le T} E\Big[\exp\Big(-\int_t^\tau r_u du\Big)(K - S^1_\tau)\big|S^1_t = S,\ X_t = X\Big]. \tag{20}$$

Write

$$\boldsymbol{D} = \boldsymbol{D}(t,S) = \big(D(t,T,S,e_1), D(t,T,S,e_2)\big)$$

$$= (D_1, D_2).$$

Then for $X_t = e_i$, $i = 1,2$, denote

$$\mathcal{C}^i = \{(S,t) \in R^+ \times [0,T] :\ D(t,T,S,e_i) > (K-S)^+\}$$

$$\mathcal{S}^i = \{(S,t) \in R^+ \times [0,T] :\ D(t,T,S,e_i) = (K-S)^+\}.$$

$\mathcal{C}^i$ is the continuation region when $X_t = e_i$ and $\mathcal{S}^i$ is the stopping region when $X_t = e_i$. For each $t \in [0,T]$ the t-section $\mathcal{C}^i_t$ of $\mathcal{C}^i$ is defined as $\{S > 0 : D(t,T,S,e_i) > (K-S)^+\}$. Following the treatment of Lemma 8.2.11 of Elliott and Kopp [4] it can be shown that $\mathcal{C}^i_t$ is an interval of the form

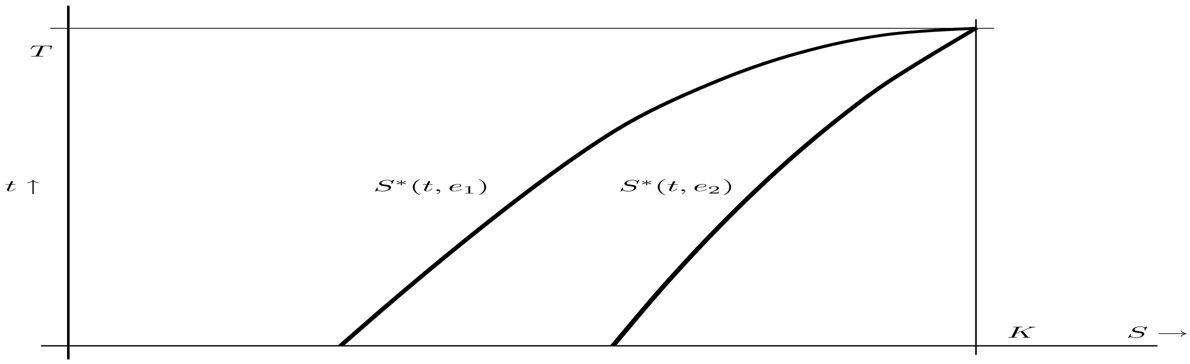
$$\big(S^*(t,e_i), \infty\big).$$

Of course, $S^*(T, e_i) = K$. However, we do not know which of $S^*(t, e_1)$, $S^*(t, e_2)$ is greater.

Assumption 3.1. We assume that

$$S^*(t, e_1) \leq S^*(t, e_2)$$

for $0 \leq t \leq T$.



Now when $S_t^1 > S^*(t, e_2)$, (S_t^1, t) is in the continuation region for both states and $D = (D_1, D_2)$ satisfies the pair of Black-Scholes equations

$$-r_1 D_1 + \frac{\partial D_1}{\partial t} + \mu_1 S \frac{\partial D_1}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 D_1}{\partial S^2} + \langle D, Ae_1 \rangle = 0 \tag{21}$$

$$-r_2 D_2 + \frac{\partial D_2}{\partial t} + \mu_2 S \frac{\partial D_2}{\partial S} + \frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 D_2}{\partial S^2} + \langle D, Ae_2 \rangle = 0. \tag{22}$$

For $S_t^1 \leq S^*(t, e_1)$

$$D_1 = D(t, T, S_t^1, e_1) = K - S \tag{23}$$

and for $S_t^1 \leq S^*(t, e_2)$

$$D_2 = D(t, T, S_t^1, e_2) = K - S. \tag{24}$$

However, for S_t^1 in the transition region between $S^*(t,e_2)$ and $S^*(t,e_2)$

$$D_1 = D(t,T,S_t^1,e_1)$$

satisfies the equation

$$-r_1 D_1 + \frac{\partial D_1}{\partial t} + \mu_1 S \frac{\partial D_1}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 D_1}{\partial S^2} + a_{11} D_1 - a_{11}(K - S) = 0. \tag{25}$$

The Common Continuation Region

We first discuss the put value function in the common continuation region where $S_t^1 > S^*(t,e_2)$ and D_1, D_2 satisfy (21) and (22).

For $S_t^1 \leq S^*(t,e_2)$

$$D_2 = D_2(t,T,S_t^1,e_2) = K - S_t^1. \tag{26}$$

On the curve $S^*(t,e_2)$ we require that D_2 satisfies a continuity condition:

$$D_2\big(t,T,S^*(t,e_2),e_2\big) = K - S^*(t,e_2) \tag{27}$$

and the smooth fit condition

$$\frac{\partial D_2}{\partial S}\big(t,T,S^*(t,e_2),e_2\big) = -1. \tag{28}$$

Clearly the American option is worth more than the European option. Write

$$\varepsilon(t,T,S,X) = D(t,T,S,X) - C(t,T,S,X)$$

for the early exercise premium. Then, because D and C satisfy the Black-Scholes equation in the common continuation region $R = \{(S,t) : S > S^*(t,e_2)\}$ so will ε.

Write $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)' = \big(\varepsilon(t,T,S,e_1), \varepsilon(t,T,S,e_2)\big)'$. Then in R

$$-r_i\varepsilon_i + \frac{\partial \varepsilon_i}{\partial t} + \mu_i S\,\frac{\partial \varepsilon_i}{\partial S} + \frac{1}{2}\,\varepsilon_i^2 S^2\,\frac{\partial^2 \varepsilon_i}{\partial S^2} + \langle \boldsymbol{\varepsilon}, Ae_i\rangle = 0 \tag{29}$$

$$\text{for} \quad i = 1, 2.$$

We shall look for a solution in separated form. That is, for $i = 1, 2$ we suppose approximately

$$\varepsilon_i(t,S) = \varepsilon(t,T,S,e_i) = f(S,e_i)g_i(t).$$

Write $\boldsymbol{f} = \boldsymbol{f}(S) = \big(f(S,e_1), f(S,e_2)\big)'$. Then we require

$$-r_i f_i g_i + f_i\,\frac{\partial g_i}{\partial t} + \mu_i S\,\frac{\partial f_i}{\partial S}\,g_i + \frac{1}{2}\,\sigma_i^2 S^2\,\frac{\partial^2 f_i}{\partial S^2}\,g_i + g_i\langle \boldsymbol{f}, Ae_i\rangle = 0. \tag{30}$$

Suppose

$$g_i(t) = E\Big[1 - \exp\Big(-\int_t^T r_u du\Big)\big|X_t = e_i\Big]$$

so

$$\frac{\partial g_i}{\partial t} = r_i\big(g_i(t) - 1\big).$$

Consequently, for $i = 1, 2$

$$-r_i f_i + \mu_i S\frac{\partial f_i}{\partial S}\,g_i + \frac{1}{2}\,\sigma_i^2 S^2\,\frac{\partial^2 f_i}{\partial S^2}\,g_i + g_i\langle \boldsymbol{f}, Ae_i\rangle = 0. \tag{31}$$

That is

$$\frac{1}{2}\,\sigma_i^2 S^2\,\frac{\partial^2 f_i}{\partial S^2} + \mu_i S\,\frac{\partial f_i}{\partial S} + \langle f_i, Ae_i\rangle = r_i\,\frac{f_i}{g_i(t)}\,. \tag{32}$$

This is not really separated as $r_i\,\frac{f_i}{g_i(t)}$ is a function of t.

In the region R we have two coupled equations for $f_1(S)$ and $f_2(S)$:

$$\frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 f_1}{\partial S^2} + \mu_1 S \frac{\partial f_1}{\partial S} + a_{11} f_1 - a_{11} f_2 = r_1 \frac{f_1}{g_1(t)} \tag{33}$$

$$\frac{1}{2}\sigma_2^2 S^2 \frac{\partial^2 f_2}{\partial S^2} + \mu_2 S \frac{\partial f_2}{\partial S} - a_{22} f_1 + a_{22} f_2 = r_2 \frac{f_2}{g_2(t)}. \tag{34}$$

We show in Appendix 2 there is a solution of the form

$$f_1(S) = \alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}$$

$$f_2(S) = \beta_1 S^{\gamma_1} + \beta_2 S^{\gamma_2}.$$

In fact

$$D(t,T,S,e_1) = C(t,T,S,e_1) + (\alpha_1 S^{\gamma_1(t)} + \alpha_2 S^{\gamma_2(t)}) g_1(t)$$

$$D(t,T,S,e_2) = C(t,T,S,e_2) + (\beta_1 S^{\gamma_1(t)} + \beta_2 S^{\gamma_2(t)}) g_2(t).$$

To complete the determination of this approximate soution we must find $\alpha_1, \alpha_2, \beta_1, \beta_2$ and the two critical boundaries $S^*(t,e_1)$, $S^*(t,e_2)$. Equations for these are derived in Appendix 2.

Transition Region

We must now consider the transition region between the two stopping curves:

$$\Gamma : \{(S,t) : S^*(t,e_1) \leq S \leq S^*(t,e_2)\}.$$

In this region $D_2 = D_2(t,T,S,e_2) = K - S$ and D_1 satisfies the Black-Scholes equation

$$-r_1 D_1 + \frac{\partial D_1}{\partial t} + \mu_1 S \frac{\partial D_1}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 D_1}{\partial S^2} + a_{11} D_1 - a_{11}(K - S) = 0. \tag{35}$$

To approximate the solution D_1 in the transition region Γ suppose the risky asset S^1 has dynamics

$$S_t^1 = S_0^1 + \int_0^t \mu_1 S_u^1 du + \int_0^t \sigma_1 S_u^1 dB_u. \tag{36}$$

However, we consider a modified instantaneous interest rate equal to $r_1 - a_{11}$ (recall $a_{11} < 0$).

Then

$$S_t^0 = \exp(r_1 - a_{11})t \tag{37}$$

and the value of a European put with strike K and exercise time T is, for $(S,t) \in \Gamma$, given by the Black-Scholes price

$$\begin{aligned}\overline{C}_1 = \overline{C}_1(S,t) &= \exp\big(-(r_1 - a_{11})(T-t)\big)E[(K - S_T^1)|S_t = S] \\ &= \exp\big(-(r_1 - a_{11})(T-t)\big)\big[SN(d_1)\exp\big(-(r_1 - a_{11})(T-t)\big) - KN(d_2)\big]\end{aligned} \tag{38}$$

where N is the standard normal distribution,

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (T-t)\left(\mu_1 + \frac{\sigma_1^2}{2}\right)}{\sigma_1\sqrt{T-t}}$$

$$\text{and} \qquad d_2 = d_1 - \sigma_1\sqrt{T-t}\,.$$

In Γ $\overline{C}_1$ satisfies the Black-Scholes equation

$$-(r_1 - a_{11})\overline{C}_1 + \frac{\partial \overline{C}_1}{\partial t} + \mu_1 S\frac{\partial \overline{C}_1}{\partial S} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 \overline{C}_1}{\partial S^2} = 0. \tag{39}$$

Define $\widetilde{C}_1(S,t) := \overline{C}_1(S,t) + \frac{a_{11}S}{(r_1-a_{11}-\mu_1)} - \frac{a_{11}K}{(r_1-a_{11})}$. Then for $(S,t) \in \Gamma$

$$-r_1\widetilde{C}_1 + \frac{\partial \widetilde{C}_1}{\partial t} + \mu_1 S\,\frac{\partial \widetilde{C}_1}{\partial S} + \frac{1}{2}\,\sigma_1^2 S^2\,\frac{\partial^2 \widetilde{C}_1}{\partial S^2}$$

$$+\, a_{11}\widetilde{C}_1 - a_{11}(K-S) = 0. \tag{40}$$

Consider $\widetilde{\varepsilon}_1 := D_1 - \widetilde{C}_1$. Then, from (35) and (40)

$$-(r_1-a_{11})\widetilde{\varepsilon}_1 + \frac{\partial \widetilde{\varepsilon}_1}{\partial t} + \mu_1 S\,\frac{\partial \widetilde{\varepsilon}_1}{\partial S} + \frac{1}{2}\,\sigma_1^2 S^2\,\frac{\partial^2 \widetilde{\varepsilon}_1}{\partial S^2} = 0. \tag{41}$$

We look for a solution for $\widetilde{\varepsilon}_1$ in separated form:

$$\widetilde{\varepsilon}_1(S,t) = F(S)G(t).$$

Then

$$-(r_1-a_{11})F(S)G(t) + F(S)\,\frac{\partial G(t)}{\partial t} + \mu_1 S\,\frac{\partial F(S)}{\partial S}\,G(t)$$

$$+\, \frac{1}{2}\,\sigma_1^2 S^2\,\frac{\partial^2 F(S)}{\partial S^2}\,G(t) = 0. \tag{42}$$

Again suppose $G(t) = 1 - e^{(r_1-a_{11})(t-T)}$ so $\frac{\partial G}{\partial t} = (r_1-a_{11})\big(G(t)-1\big)$. Then

$$-(r_1-a_{11})F(S) + \mu_1 S\,\frac{\partial F}{\partial S}\,G(t) + \frac{1}{2}\,\sigma_1^2 S^2\,\frac{\partial^2 F}{\partial S^2}\,G(t) = 0 \tag{43}$$

or

$$\frac{1}{2}\,\sigma_1^2 S^2\,\frac{\partial^2 F}{\partial S^2} + \mu_1 S\,\frac{\partial F}{\partial S} + (a_{11}-r_1)\,\frac{F(S)}{G(t)} = 0.$$

This is not really in separated form as $\frac{(a_{11}-r_1)}{G(t)}$ is a function of t.

Again we look for a solution in the form:

$$F(S) = A_1 S^{\lambda_1} + A_2 S^{\lambda_2}. \tag{44}$$

In Appendix 3 we show there is an approximate solution of the form

$$D_1(t,T,S,e_1) = \overline{C}_1(t,T,S,e_1) + \frac{a_{11}S}{(r_1 - a_{11} - \mu_1)} - \frac{a_{11}K}{(r_1 - a_{11})}$$
$$+ \big(A_1 S^{\lambda_1(t)} + A_2 S^{\lambda_2(t)}\big)G(t).$$

Further, equations are obtained which determine $A_1 A_2$ and the free boundaries $S^*(t,e_1)$, $S^*(t,e_2)$.

4. Conclusion

We have considered an economy which can have finitely many states. These are identified with the standard unit vectors $e_i = (0,\ldots,0,1,0,\ldots,0)' \in R^N$ and the transition process is modeled as a Markov chain X. The Black-Scholes market is extended to the case where the parameters depend on X. The characteristic function of the occupation time is derived and used to price European options. Finally, the approximate value of American options due to Barone-Adesi and Whaley is generalized to this regime-switching framework.

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APPENDIX 1

Lemma A.1.

$$E\Big[\exp i\Big(\sum_{j=1}^{N-1}\theta_j T_j\Big)\Big] = \langle \exp[(A + i\,\mathrm{diag}\,\boldsymbol{\theta})T]X_0, \mathbf{1}\rangle$$

where $\mathbf{1} = (1,1,\ldots,1)' \in R^N$.

Proof. Consider the R^N valued process

$$Z_t := \exp i\Big(\int_0^t \langle \boldsymbol{\theta}, X_u\rangle\, du\Big)X_t.$$

Then

$$\begin{aligned} dZ_t &= \exp i\Big(\int_0^t \langle \boldsymbol{\theta}, X_u\rangle du\Big)\cdot dX_t + \exp i\Big(\int_0^t \langle \boldsymbol{\theta}, X_u\rangle du\Big)\cdot i\langle \boldsymbol{\theta}, X_t\rangle X_t dt \\ &= (A + i\,\mathrm{diag}\,\boldsymbol{\theta})Z_u du + \exp i\Big(\int_0^t \langle \boldsymbol{\theta}, X_u\rangle du\Big)dM_t \end{aligned}$$

where the dynamics of X are given by (1). Therefore,

$$Z_t = X_0 + \int_0^t (A + i\,\mathrm{diag}\,\boldsymbol{\theta})Z_u du + \int_0^t \exp\Big(i\int_0^u \langle \boldsymbol{\theta}, X_u\rangle du\Big)dM_u.$$

The final integral is a martingale, so taking expected values

$$E[Z_t] = X_0 + \int_0^t (A + i\,\mathrm{diag}\,\boldsymbol{\theta})E[Z_u]du$$

$$\text{and} \quad E[Z_t] = \exp[(A + i\,\mathrm{diag}\,\boldsymbol{\theta})t]\cdot X_0.$$

$E[Z_t]$ is an R^N valued process and the characteristic function

$$E\Big[\exp i\Big(\int_0^T \langle \boldsymbol{\theta}, X_u\rangle du\Big)\Big]$$

is obtained by summing its components. That is,

$$E\Big[\exp i\Big(\int_0^T \langle\boldsymbol{\theta}, X_u\rangle du\Big)\Big] = E\Big[\Big\langle \exp i\Big(\int_0^T \langle\theta, X_u\rangle du\Big) X_T, \mathbf{1}\Big\rangle\Big]$$

$$= \langle \exp\left[(A + i\,\mathrm{diag}\,\boldsymbol{\theta})T\right]X_0, \mathbf{1}\rangle.$$

Remark 2. When $N = 2$, that is X switches between only 2 states, the equation for the characteristic function of T_1 reduces to an ordinary differential equation. The inverse Fourier transform gives rise to a density function for T_1 in terms of Bessel processes. See Guo [5], [6].

In fact, for $X_t \in \{e_1, e_2\}$, $e_1 = (0,1)'$, $e_2 = (0,1)'$ write:

$$Z_t^1 := \exp\Big(i\int_0^t \theta_1\langle e_1, X_u\rangle du\Big)\,\langle e_1, X_t\rangle$$

$$Z_t^2 := \exp\Big(i\int_0^t \theta_1\langle e_1, X_u\rangle du\Big)\,\langle e_2, X_t\rangle\,.$$

Then:

$$Z_t^1 = \langle e_1, X_0\rangle + \int_0^t (i\theta_1 + a_{11})Z_u' du$$

$$- a_{22}\int_0^t Z_u^2 du + \int_0^t \exp\Big(i\int_0^u \theta_1\langle e_1, X_s\rangle ds\Big)\langle e_1, dM_u\rangle$$

$$\text{and}\quad Z_t^2 = \langle e_2, X_0\rangle - \int_0^t (a_{11})Z_u^1 du$$

$$- a_{22}\int_0^t Z_u^2 du + \int_0^t \exp\Big(i\int_0^u \theta_1\langle e_1, X_s\rangle\Big)\langle e_2, dM_u\rangle.$$

The final integrals in Z^1 and Z^2 are martingales. Writing

$$\widehat{Z}_t^1 = E[Z_t^1], \quad \widehat{Z}_t^2 = E[Z_t^2]$$

and taking expectations we have

$$\widehat{Z}_t^1 = \langle e_1, X_0\rangle + \int_0^t (i\theta_1 + a_{11})\widehat{Z}_u^1 du - a_{22}\int_0^t \widehat{Z}_u^2 du \tag{A1}$$

$$\widehat{Z}_t^2 = \langle e_2, X_0\rangle - \int_0^t a_{11}\widehat{Z}_u^1 du + a_{22}\int_0^t \widehat{Z}_u^2 du. \tag{A2}$$

Therefore,

$$\widehat{Z}_t^2 = e^{a_{22}t}\Big(\langle e_2, X_0\rangle - a_{11}\int_0^t e^{-a_{22}u}\widehat{Z}_u^1 du\Big).$$

Substituting $\widehat{Z}_t^2$ in (A1)

$$\widehat{Z}_t^1 = \langle e_1, X_0\rangle + \int_0^t (i\theta_1 + a_{11})\widehat{Z}_u^1 du - a_{22}\langle e_2, X_0\rangle\int_0^t e^{a_{22}u} du$$

$$+ a_{22}a_{11}\int_0^t e^{a_{22}u}\Big(\int_0^u e^{-a_{22}s}\widehat{Z}_s^1 ds\Big) du. \tag{A3}$$

Write $Y_t^1 = e^{-a_{22}t}\widehat{Z}_t^1$. Then from (A3)

$$\frac{dY_t^1}{dt} = (a_{11} - a_{22} + i\theta_1)Y_t^1 - a_{22}\langle e_2, X_0\rangle + a_{22}a_{11}\int_0^t Y_u^1 du.$$

Consequently

$$\frac{d^2Y_t^1}{dt^2} - (a_{11} - a_{22} + i\theta_1)\frac{dY_t^1}{dt} - a_{22}a_{11}Y_t^1 = 0$$

with

$$Y_0^1 = \widehat{Z}_0^1 = \langle e_1, X_0\rangle \tag{A4}$$

$$\text{and}\quad \frac{dY^1}{dt}\Big|_{t=0} = (a_{11} - a_{22} + i\theta_1)\langle e_1, X_0\rangle - a_{22}\langle e_2, X_0\rangle. \tag{A5}$$

Suppose y_1 and y_2 are the roots of the quadratic equation $y^2 - (a_{11} - a_{22} + i\theta_1)y - a_{22}a_{11}y = 0$. Then

$$Y_t^1 = c_1e^{y_1t} + c_2e^{y_2t}$$

where c_1 and c_2 are determined by the boundary conditions (A4) and (A5) so that $c_1 + c_2 = \langle e_1, X_0\rangle$ and $c_1 y_1 + c_2 y_2 = (a_{11} - a_{22} + i\theta_1)\langle e_1, X_0\rangle - a_{22}\langle e_2, X_0\rangle$. Now

$$\widehat{Z}_t^1 = e^{a_{22}t} Y_t^1 = c_1 e^{(a_{22}+y_1)t} + c_2 e^{(a_{22}+y_2)t}$$

$$\text{and}\quad \widehat{Z}_t^2 = e^{a_{22}t}\Big(\langle e_2, X_0\rangle - a_{11}\int_0^t \big(c_1 e^{y_1 u} + c_2 e^{y_2 u}\big)du\Big)$$
$$= e^{a_{22}t}\langle e_2, X_0\rangle - a_{11}e^{a_{22}t}\Big[\frac{c_1}{y_1}\left(e^{y_1 t} - 1\right) + \frac{c_2}{y_2}\left(e^{y_2 t} - 1\right)\Big].$$

Finally the characteristic function

$$E\Big[\exp\Big(i\int_0^t \theta_1\langle e_1, X_u\rangle du\Big)\Big] = \widehat{Z}_t^1 + \widehat{Z}_t^2$$
$$= e^{a_{22}t}\Big[\langle e_2, X_0\rangle - a_{22}^{-1}\big((a_{11} - a_{22} + i\theta_1)\langle e_1, X_0\rangle - a_{22}\langle e_2, X_0\rangle\big)$$
$$\times \frac{c_1}{y_1}(y_1 - a_{11})e^{y_1 t} + \frac{c_2}{y_2}(y_2 - a_{11})e^{y_2 t}\Big].$$

Appendix 2

We consider equations (38) and (39) and look for a solution of the form:

$$f_1(S) = \alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}$$

$$f_2(S) = \beta_1 S^{\gamma_1} + \beta_2 S^{\gamma_2}.$$

Then

$$\frac{1}{2}\sigma_1^2\big(\alpha_1\gamma_1(\gamma_1 - 1)S^{\gamma_1} + \alpha_2\gamma_2(\gamma_2 - 1)S^{\gamma_2}\big)$$
$$+ \mu_1(\alpha_1\gamma_1 S^{\gamma_1} + \alpha_2\gamma_2 S^{\gamma_2}) + a_{11}(\alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}) - a_{11}(\beta_1 S^{\gamma_1} + \beta_2 S^{\gamma_2})$$
$$= \frac{r_1}{g_1(t)}\left(\alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}\right)$$

and

$$\frac{1}{2}\,\sigma_2^2\big(\beta_1\gamma_1(\gamma_1-1)S^{\gamma_1}+\beta_2\gamma_2(\gamma_2-1)S^{\gamma_2}\big)$$
$$+\,\mu_2(\beta_1\gamma_1 S^{\gamma_1}+\beta_2\gamma_2 S^{\gamma_2})-a_{22}(\alpha_1 S^{\gamma_1}+\alpha_2 S^{\gamma_2})+a_{22}(\beta_1 S^{\gamma_1}+\beta_2 S^{\gamma_2})$$
$$=\frac{r_2}{g_2(t)}\,(\beta_2 S^{\gamma_1}+\beta_2 S^{\gamma_2}).$$

Equating coefficients of S^{γ_1} and S^{γ_2} gives four equations:

$$\frac{1}{2}\,\sigma_1^2\alpha_1\gamma_1(\gamma_1-1)+\mu_1\alpha_1\gamma_1+a_{11}\alpha_1-a_{11}\beta_1-\frac{r_1}{g_1(t)}\,\alpha_1=0 \tag{A6}$$

$$\frac{1}{2}\,\sigma_1^2\alpha_2\gamma_2(\gamma_2-1)+\mu_1\alpha_2\gamma_2+a_{11}\alpha_2-a_{11}\beta_2-\frac{r_1}{g_1(t)}\,\alpha_2=0 \tag{A7}$$

$$\frac{1}{2}\,\sigma_2^2\beta_1\gamma_1(\gamma_1-1)+\mu_2\beta_1\gamma_1-a_{22}\alpha_1+a_{22}\beta_1-\frac{r_2}{g_2(t)}\,\beta_1=0 \tag{A8}$$

$$\frac{1}{2}\,\sigma_2^2\beta_2\gamma_2(\gamma_2-1)+\mu_2\beta_2\gamma_2-a_{22}\alpha_2+a_{22}\beta_1-\frac{r_2}{g_2(t)}\,\beta_2=0. \tag{A9}$$

From (A6)

$$a_{11}\,\frac{\beta_1}{\alpha_1}=\frac{1}{2}\,\sigma_1^2\gamma_1(\gamma_1-1)+\mu_1\gamma_1+a_{11}-\frac{r_1}{g_1(t)}\,. \tag{A10}$$

From (A8)

$$a_{22}\,\frac{\alpha_1}{\beta_1}=\frac{1}{2}\,\sigma_2^2\gamma_1(\gamma_1-1)+\mu_2\gamma_1+a_{22}-\frac{r_2}{g_2(t)}\,. \tag{A11}$$

Therefore, γ_1 should be a solution of the fourth order equation

$$a_{11}a_{22}=\Big(\frac{1}{2}\,\sigma_1^2\gamma(\gamma-1)+\mu_1\gamma+a_{11}-\frac{r_1}{g_1(t)}\Big)\Big(\frac{1}{2}\,\sigma_2^2\gamma(\gamma-1)+\mu_2\gamma+a_{22}-\frac{r_2}{g_2(t)}\Big).$$

Similarly γ_2 satisfies the same equation. Note $g_1(t)$ and $g_2(t)$ are functions of t. Therefore, $\gamma_1(t),\gamma_2(t)$ are also functions of t.

Consider the function

$$F(\gamma) = \Big(\frac{1}{2}\sigma_1^2\gamma(\gamma-1) + \mu_1\gamma + a_{11} - \frac{r_1}{g_1(t)}\Big)$$
$$\times\Big(\frac{1}{2}\sigma_2^2\gamma(\gamma-1) + \mu_2\gamma + a_{22} - \frac{r_2}{g_2(t)}\Big) - a_{11}a_{22}. \qquad \text{(A12)}$$

Adapting the argument of Guo [5], [6] we observe: $F(+\infty) > 0, \quad F(-\infty) = 0$

$$F(0) = \frac{r_1 r_2}{g_1(t)g_2(t)} - a_{11}\frac{r_2}{g_2(t)} - a_{22}\frac{r_1}{g_1(t)} > 0$$

because $a_{11} < 0, \quad a_{22} < 0$. Let ρ_1, ρ_2 be the roots of the quadratic equation

$$G(\rho) = \frac{1}{2}\sigma_1^2\rho(\rho-1) + \mu_1\rho + a_{11} - \frac{r_1}{g_1(t)} = 0.$$

Then $G(\infty) = +\infty, \quad G(-\infty) = +\infty$ and $G(0) = a_{11} - \frac{r_1}{g_1(t)} < 0$ as $a_{11} < 0$. Therefore, $G(\rho)$ has two real roots

$$\rho_1 > 0, \quad \rho_1 < 0.$$

Now $F(\rho_i) = -a_{11}a_{22} < 0$. Consequently $F(\gamma) = 0$ has four real roots, two of which are positive and two negative. Because the American put has value 0 when $S = +\infty$ our solution cannot involve any positive power of S. Write $\gamma_1(t), \gamma_2(t)$ for the two negative, (time varying) roots of $F(\gamma) = 0$. Then (approximately),

$$\varepsilon_1(t,T,S,e_1) = (\alpha_1 S^{\gamma_1(t)} + \alpha_2 S^{\gamma_2(t)})g_1(t)$$

$$\varepsilon_2(t,T,S,e_2) = (\beta_1 S^{\gamma_1(t)} + \beta_2 S^{\gamma_2(t)})g_2(t)$$

and the approximate American put value is, for $i = 1, 2$

$$D(t,T,S,e_1) = C(t,T,S,e_i) + \varepsilon_i(t,T,S,e_i).$$

That is:

$$D(t,T,S,e_1) = C(t,T,S,e_1) + (\alpha_1 S^{\gamma_1(t)} + \alpha_2 S^{\gamma_2(t)}) g_1(t)$$

$$D(t,T,S,e_2) = C(t,T,S,e_2) + (\beta_1 S^{\gamma_1(t)} + \beta_2 S^{\gamma_2(t)}) g_2(t).$$

To complete the determination of this approximate solution we must find $\alpha_1, \alpha_2, \beta_1, \beta_2$ and the two critical boundaries $S^*(t,e_1)$, $S^*(t,e_2)$. From (A6) above we have for $i = 1,2$

$$\beta_i = a_{ii}^{-1}\Big(\frac{1}{2}\,\sigma_i^2 \gamma_i(t)\big(\gamma_i(t)-1\big) + \mu_i \gamma_i(t) + a_{ii} - \frac{r_i}{g_i(t)}\Big)\alpha_i. \tag{A13}$$

That is, $\alpha_i = \lambda_i^{-1}\beta_i$ where $\lambda_i = a_{ii}^{-1}\big(\frac{1}{2}\,\sigma_i^2\gamma_i(t)(\gamma_i(t)-1) + \mu_i\gamma_i(t) + a_{ii} - \frac{r_i}{g_i(t)}\big)$.

On the boundary $S^*(t,e_2)$

$$D\big(t,T,S^*(t,e_2),e_2\big) = C\big(t,T,S^*(t,e_2),e_2\big) + \big(\beta_1 S^*(t,e_2)^{\gamma_1(t)} + \beta_2 S^*(t,e_2)^{\gamma_2(t)}\big) g_2(t)$$

$$= K - S^*(t,e_2). \tag{A14}$$

The smooth fit condition for D on $S^*(t,e_2)$ gives:

$$\frac{\partial D}{\partial S}\big(t,T,S^*(t,e_2),e_2\big) = \frac{\partial C}{\partial S}\big(t,T,S^*(t,e_2),e_2\big)$$

$$+ \big(\beta_1\gamma_1(t) S^*(t,e_2)^{\gamma_1(t)-1} + \beta_2\gamma_2(t) S^*(t,e_2)^{\gamma_2(t)-1}\big) g_2(t)$$

$$= -1. \tag{A15}$$

The equations, (A14) and (A15) give, (omitting the 't' variable), and writing $S_i^* = S^*(t,e_i)$, $i = 1,2$

$$\beta_1 S_2^{*\gamma_1} g_2(\gamma_2 - \gamma_1) = S_2^*\Big(1 + \frac{\partial C_2}{\partial S}\Big) + \gamma_2(K - S_2^* - C_2)$$

$$\beta_2 S_1^{*\gamma_2} g_2(\gamma_1 - \gamma_2) = S_2^*\Big(1 + \frac{\partial C_2}{\partial S}\Big) + \gamma_2(K - S_2^* - C_2).$$

Therefore,

$$\beta_1 = \Big[S_2^*\Big(1 + \frac{\partial C_2}{\partial S}\Big) + \gamma_2(K - S_2^* - C_2)\Big]\Big[S_2^{*\gamma_1} g_2(\gamma_2 - \gamma_1)\Big]^{-1}$$

$$\beta_2 = \Big[S_2^*\Big(1 + \frac{\partial C_2}{\partial S}\Big) + \gamma_1(K - S_2^* - C_2)\Big]\Big[S_2^{*\gamma_2} g_2(\gamma_1 - \gamma_2)\Big]^{-1}.$$

Appendix 3

Consider the equation

$$\frac{1}{2}\,\sigma_1^2 S^2\,\frac{\partial^2 F}{\partial S^2} + \mu_1 S\,\frac{\partial F}{\partial S} + (a_{11} - r_1)\,\frac{F(S)}{G(t)} = 0.$$

We shall show there is a solution of the form

$$F(S) = A_1 S^{\lambda_1} + A_2 S^{\lambda_2}.$$

For this λ_1 and λ_2 must be time dependent roots of

$$\frac{\sigma_1^2}{2}\lambda(\lambda - 1) + \mu_1\lambda - \frac{(r_1 - a_{11})}{G(t)} = 0. \tag{A16}$$

Note $\lambda_1(t) < 0 < \lambda_2(t)$.

Therefore, we propose as an approximate solution for D_1 in the transition region Γ the function

$$D_1(t,T,S,e_1) = \overline{C}_1(t,T,S,e_1) + \frac{a_{11}S}{(r_1 - a_{11} - \mu_1)} - \frac{a_{11}K}{(r_1 - a_{11})} + \big(A_1 S^{\lambda_1(t)} + A_2 S^{\lambda_2(t)}\big)G(t). \tag{A17}$$

Continuity on $S^*(t, e_1)$ gives:

$$
\begin{aligned}
D_1(t, T, S, e_1) &= K - S^*(t, e_1) \\
&= \overline{C}_1\big(t, T, S^*(t, e_1), e_1\big) + \frac{a_{11} S^*(t, e_1)}{(r_1 - a_{11} - \mu_1)} - \frac{a_{11} K}{(r_1 - a_{11})} \\
&\quad + \big(A_1 S^*(t, e_1)^{\lambda_1(t)} + A_2 S^*(t, e_1)^{\lambda_2(t)}\big) G(t). \qquad \text{(A18)}
\end{aligned}
$$

Smoothness on $S^*(t, e_1)$ gives:

$$
\begin{aligned}
\frac{\partial D_1}{\partial S}\big(t, T, S^*(t, e_1), e_1\big) &= \frac{\partial \overline{C}}{\partial S}\big(t, T, S^*(t, e_1), e_1\big) + \frac{a_{11}}{(r_1 - a_{11} - \mu_1)} \\
&\quad + \big(A_1 \lambda_1(t) S^*(t, e_1)^{\lambda_1(t)-1} + A_2 \lambda_2(t) S^*(t, e_1)^{\lambda_2(t)-1}\big) G(t) \\
&= -1. \qquad \text{(A19)}
\end{aligned}
$$

Writing $S^*(t, e_1) = S_1^*$, $\lambda_1(t) = \lambda_1$, $\lambda_2(t) = \lambda_2$ and $\overline{C}_1\big(t, T, S^*(t, e_1), e_1\big) = \overline{C}_1$ these equations give for each t

$$
\begin{aligned}
A_1 &= A_1(t) \\
&= \Big[\frac{(r_1 - \mu_1)(\lambda_2 - 1) S_1^*}{(r_1 - a_{11} - \mu_1)} - \frac{r_1 \lambda_2 K}{(r_1 - a_{11})} - \frac{\partial \overline{C}_1}{\partial S} \cdot S_1^* + \lambda_2 \overline{C}_1\Big] \Big/ G(t)(\lambda_1 - \lambda_2)(S_1^*)^{\lambda_1} \qquad \text{(A20)}
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= A_2(t) \\
&= \Big[\frac{(r_1 - \mu_1)(\lambda_1 - 1) S_1^*}{(r_1 - a_{11} - \mu_1)} - \frac{r_1 \lambda_1 K}{(\sigma_1 - a_{11})} - \frac{\partial \overline{C}_1}{\partial S} \cdot S_1^* + \lambda_1 \overline{C}_1\Big] \Big/ G(t)(\lambda_2 - \lambda_1)(S_1^*)^{\lambda_2}. \qquad \text{(A21)}
\end{aligned}
$$

Finally, we use the continuity and smoothness of D_1 at the boundary $S_2^* = S^*(t, e_2)$.

This gives

$$\big(A_1(S_2^*)^{\lambda_1} + A_2(S_2^*)^{\lambda_2}\big)G(t) + \overline{C}_1(t,T,S_2^*,e_1) + \frac{a_{11}S_2^*}{(r_1 - a_{11} - \mu_1)} - \frac{a_{11}K}{(r_1 - a_{11})}$$
$$= C_1(t,T,S_2^*,e_1) + \big(\alpha_1(S_2^*)^{\gamma_1} + \alpha_2(S_2^*)^{\gamma_2}\big)g(t) \tag{A22}$$

and

$$\big(A_1\lambda_1(S^*)2)^{\lambda_1-1} + A_2\lambda_2(S_2^*)^{\lambda_2-1}\big)G(t) + \frac{\partial\overline{C}_1}{\partial S}(t,T,S_2^*,e_1) + \frac{a_{11}}{(r_1 - a_{11} - \mu_1)}$$
$$= \frac{\partial C_1}{\partial S}(t,T,S_2^*,e_1) + \big(\alpha_1\gamma_1(S_2^*)^{\gamma_1-1} + \alpha_2\gamma_2(S_2^*)^{\gamma_2-1}\big)g(t). \tag{A23}$$

These two equations, (A22) and (A23), can be solved numerically to determine the free boundaries $S^*(t,e_1)$ and $S^*(t,e_2)$.