# Q-Options and Dual-Expiry Exotics

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**Peter W. Buchen**
**School of Mathematics and Statistics**
**University of Sydney, Australia**
**peterb@maths.usyd.edu.au**

## 1 Introduction

There exists a class of exotic options that depend on a single underlying asset but whose payoff structure involves two fixed future dates, labelled $T_0$ and $T_1$ with $T_0 < T_1$ throughout this paper. Typically the option holder makes a decision at time $T_0$ that affects the payoff at time $T_1$. This may include the possibility of accepting an immediate payoff at time $T_0$ with the option lapsing worthless thereafter. Options belonging to this class are typically exotic or non-standard and may sensibly be referred to as dual-expiry options. In the standard Black-Scholes (BS) framework, when the underlying asset price follows a continuous geometrical Brownian motion, the arbitrage free prices of these dual-expiry options invariably involve the bivariate normal distribution function. Examples of dual-expiry exotics include: compound options, chooser options, partial barrier options, cliquet options, extendable options and even American call options on an asset with a single known dividend payment.

This paper develops a theory for pricing all these options in terms of a family of elementary binary options. A binary option is one that pays on its expiry date: an agreed amount if the underlying asset price is above (or below) a fixed exercise price and zero otherwise. Binary options on the underlying asset and on cash are called asset binaries and bond binaries respectively and are examples of first order binaries. Binary options on first order binaries are examples of second order (or compound) binaries. In this paper, all exotic option prices can be readily expressed in terms of certain linear combinations of first and second order binaries. These linear combinations, which are generalisations of familiar calls and puts, are the $Q-$options referred to in the title.

Because the formulas derived in this paper are expressed in terms of the prices of more elementary contracts, they are generally valid for a wider class of models than that of the standard BS framework. While closed-form analytic formulas are readily given in the BS world, this may not be possible in more complicated models of the underlying asset price.

It is well known that explicit BS formulas for exotic options can be quite daunting. For example, one such formula for a partial-time-end-out barrier option is the sum of twelve bivariate normals, each with a complicated set of arguments (Heynen and Kat 1994). A great advantage of the method and notation used in this paper is that such expressions have a considerably simpler mathematical representation. This makes for greater transparency of the solution, easier derivation of new exotics of the same class and reduced error in computation.

# 2 Theory and Notation

Consider an underlying asset (henceforth referred to as the 'stock') whose price $x$ at time $t$ follows a continuous Markov process described by some Itô stochastic differential equation. In such circumstances the price $V_0(x,t)$ of an option or derivative on the stock with $T$ expiry payoff $f(x)$ satisfies a diffusion pde $\mathcal{L}V_0(x,t)=0$ for $t<T$ and with expiry condition $V_0(x,T)=f(x)$. In the familiar Black-Scholes framework, the above pde is the usual BS-pde

$$\mathcal{L}V_0 = \frac{\partial V_0}{\partial t} - rV_0 - rx\frac{\partial V_0}{\partial x} - \tfrac{1}{2}\sigma^2x^2\frac{\partial^2 V_0}{\partial x^2} = 0 \tag{1}$$

where $r,\sigma$ are the risk-free rate and stock volatility respectively. Option price $V_0(x,t)$ will be called a 'standard' option.

An up-binary option of exercise price $\xi$ on the standard option just defined is a contract with expiry payoff $f(x)$ if $x>\xi$ and zero otherwise. A down-binary option pays $f(x)$ at expiry only if $x<\xi$ and otherwise zero. Let $s=(+,-)$ be sign indicators for the up and down binaries respectively. Then the payoff function for these binaries can be written in the form

$$V_\xi^s(x,T) = f(x)\mathbf{1}(sx>s\xi). \tag{2}$$

**Up-Down Parity**
If $V_0(x,t)$ denotes the time $t$ price of the standard option, then the corresponding up and down binaries satisfy the parity relation

$$V_\xi^+(x,t) + V_\xi^-(x,t) = V_0(x,t) \tag{3}$$

independent of the exercise price $\xi$.

**Asset and Bond Binaries**
If $f(x) = x$, the standard option simply pays one unit of the asset at expiry and obviously $V_0(x,t) = x$ for all $t$. The corresponding binaries will be termed asset binaries (also known as asset-or-nothing binaries) and will be represented by the notation $A_\xi^s(x,\tau)$ with $\tau = T - t$. Similarly, if $f(x) = 1$, the standard option pays one unit of cash at expiry and $V_0(x,t) = e^{-rt}$ independent of $x$. Its binaries are termed bond binaries (also known as cash-or-nothing binaries) and will be designated by $B_\xi^s(x,\tau)$. These asset and bond binaries are fundamental building blocks for the class of exotic options considered in this article. Their up-down parity relations are respectively

$$A_\xi^+(x,\tau) + A_\xi^-(x,\tau) = x \qquad \text{and} \qquad B_\xi^+(x,\tau) + B_\xi^-(x,\tau) = e^{-r\tau}. \tag{4}$$

**Theorem 1** An option with expiry payoff $V_\xi^s(x,T) = (hx+k)\mathbf{1}(sx > s\xi)$ (that is pays $h$ units of stock and $k$ unis of cash) has arbitrage free value $V_\xi^s(x,t) = hA_\xi^s(x,t) + kB_\xi^s(x,t)$ at time $t < T$.

**First Order Q-Option**
Consider a standard contract that pays $f(x) = s(x-k)$ at expiry. The prices of corresponding up and down binaries are then given by

$$\boxed{Q_\xi^s(x,\tau;k) = s[A_\xi^s(x,\tau) - kB_\xi^s(x,\tau)]} \tag{5}$$

With this definition it is clear that $Q_\xi^+(x,\tau;k)$ denotes the price of a generalised European call option and $Q_\xi^-(x,\tau;k)$ that of a generalised European put option. These options are generalised in the sense that their exercise price $\xi$ is different from their strike price $k$. If $\xi = k$, then they reduce to standard European calls and puts. The associated up-down parity is in fact a put-call parity relation:

$$Q_\xi^+(x,\tau;k) - Q_\xi^-(x,\tau;k) = x - ke^{-r\tau} \tag{6}$$

Note that the $Q-$option price define by Eq(5) has a positive payoff for all $x$ at time $T$ if $s\xi \geq sk$.

**Black-Scholes Expressions**
The following expressions are well known for the case of the BS-framework:

$$\begin{aligned} A_\xi^s(x,\tau) &= x\mathcal{N}(sd_\xi) && (7)\\ B_\xi^s(x,\tau) &= e^{-r\tau}\mathcal{N}(sd'_\xi) && (8)\\ Q_\xi^s(x,\tau;k) &= s\left[x\mathcal{N}(sd_\xi) - ke^{-r\tau}\mathcal{N}(sd'_\xi)\right] && (9)\end{aligned}$$

where

$$[d_\xi, d'_\xi](x,\tau) = \frac{\log(x/\xi) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \qquad \tau = T - t. \qquad (10)$$

**Second Order Binaries**
Consider times $t < T_0 < T_1$, where $t$ denotes the current time, $T_0$ is the expiry date of a second order binary option of exercise price $\xi_0$ and $T_1$ is the expiry date of an underlying first order binary option of exercise price $\xi_1$. Specifically, the payoff at time $T_0$ is assumed to be

$$V(x, T_0) = F_{\xi_1}^{s_1}(x,\tau)\mathbf{1}(s_0 x > s_0\xi_0) \qquad (11)$$

where $s_0, s_1$ depict up and down binaries at times $T_0$ and $T_1$ respectively, $\tau = T_1 - T_0$ and $F$ denotes either $A$ for an underlying asset binary, $B$ for an underlying bond binary or $Q$ for an underlying first order $Q-$binary. The prices of these second order binary options prior to $T_0$ will be designated respectively by

$$A_{\xi_0\xi_1}^{s_0s_1}(x,t_0,t_1); \qquad B_{\xi_0\xi_1}^{s_0s_1}(x,t_0,t_1) \qquad (12)$$

and

$$\boxed{Q_{\xi_0\xi_1}^{s_0s_1}(x,t_0,t_1;k) = s_1[A_{\xi_0\xi_1}^{s_0s_1}(x,t_0,t_1) - kB_{\xi_0\xi_1}^{s_0s_1}(x,t_0,t_1)]} \qquad (13)$$

where $t_0 = T_0 - t$ and $t_1 = T_1 - t$. Note that the strike price $k$ in the second order $Q-$option is effective only at time $T_1$.

Obviously for fixed exercise prices $\xi_i$ and fixed expiry dates $T_i$ there are four such second order binaries corresponding to $s_i = (+,-)$. In addition, from Eq(3), the following up-down parity relation exists:

$$F_{\xi_0\xi_1}^{+s_1}(x,t_0,t_1) + F_{\xi_0\xi_1}^{-s_1}(x,t_0,t_1) = F_{\xi_1}^{s_1}(x,t_1) \qquad (14)$$

where once agin $F$ denotes either $A, B$ or $Q$. Second order binary options are also called compound binaries.

**Black-Scholes Expressions**
BS expressions for the second order binaries involve the bivariate normal distribution function $\mathcal{N}(x, y; \rho)$,where $\rho$ is the correlation coefficient. Specifically, it can be shown (for example, by risk neutral expectations) that

$$A^{s_0 s_1}_{\xi_0 \xi_1}(x, t_0, t_1) = x\mathcal{N}(s_0 d_{\xi_0}, s_1 d_{\xi_1}; s_0 s_1 \rho) \tag{15}$$

$$B^{s_0 s_1}_{\xi_0 \xi_1}(x, t_0, t_1) = e^{-rt_1}\mathcal{N}(s_0 d'_{\xi_0}; s_1 d'_{\xi_1}, s_0 s_1 \rho) \tag{16}$$

where

$$[d_{\xi_i}, d'_{\xi_i}](x, t_i) = \frac{\log(x/\xi_i) + (r \pm \frac{1}{2}\sigma^2)t_i}{\sigma\sqrt{t_i}} \qquad (i = 0, 1) \tag{17}$$

$$\text{and} \qquad \rho = \sqrt{t_0/t_1}. \tag{18}$$

The BS expression for the second order $Q-$option $Q^{s_0 s_1}_{\xi_0 \xi_1}(x, t_0, t_1; k)$ is then given by Eq(13).

# 3 Applications

The key to pricing dual-expiry options is the observation that a time $T_0$ payoff of the form of Eq(11) (that is, a first-order binary multiplied by an indicator function) leads to an arbitrage free price of the form of a second order bianry: $V(x, t) = F^{s_0 s_1}_{\xi_0 \xi_1}(x, t_0, t_1)$ at time $t < T_0$. This simple principle is used in deriving all the dual-expiry exotics considered in the ensuing applications.

## 3.1 Compound Options

There are four basic compound options (Geske 1979a) commonly referred to as a call-on-call, call-on-put, put-on-call and put-on-put. At time $T_0$ the holder of a compound option has the right to buy (or sell) for price $k_0$ a standard European call (or put) option of strike price $b$ say. Let $\alpha, \beta = (+, -)$ for a call or put at times $T_0$ and $T_1$. Then an $\alpha\beta-$compound option has $T_0$

payoff given by

$$
\begin{aligned}
V_{\alpha\beta}(x, T_0) &= \left[\alpha\left(Q_b^{\beta}(x, \tau; b) - k_0\right)\right]^+ \\
&= \alpha\left[Q_b^{\beta}(x, \tau; b) - k_0\right] \mathbf{1}(\gamma x > \gamma a)
\end{aligned}
\tag{19}
$$

where $\gamma = \alpha\beta$ and $x = a$ is the unique root of the equation

$$
Q_b^{\beta}(x, \tau; b) = k_0 \tag{20}
$$

The root $x = a$ (dependent on $\beta$) is unique because $Q_b^+(x, \tau; b)$ (a standard European call) is a monotonic increasing function of $x$, and $Q_b^-(x, \tau; b)$ (a standard European put) is a monotonic decreasing function of $x$. It follows from the theory of first and second order binaries presented previously that for any time $t < T_0 < T_1$, the price of the compound option is given by the expression

$$
\boxed{V_{\alpha\beta}(x, t) = \alpha Q_{ab}^{\gamma\beta}(x, t_0, t_1; b) - \alpha k_0 B_a^{\gamma}(x, t_0)} \tag{21}
$$

involving a second order $Q-$option and a first order bond binary option.

## 3.2 Chooser Options

A complex chooser option (Rubinstein 1991) gives its holder the right at $T_0$ to select either a call of strike price $a_1$ and expiry $T_1$ or a put of strike price $a_2$ and expiry $T_2$. It is assumed that $T_0 < \min(T_1, T_2)$. A simple chooser option has the property that $a_1 = a_2$ and $T_1 = T_2$. The time $T_0$ payoff of the complex chooser is

$$
\begin{aligned}
V(x, T_0) &= \max\left[Q_{a_1}^+(x, \tau_1; a_1),\ Q_{a_2}^-(x, \tau_2; a_2)\right] \\
&= Q_{a_1}^+(x, \tau_1; a_1)\mathbf{1}(x > a_0) + Q_{a_2}^-(x, \tau_2; a_2)\mathbf{1}(x < a_0)
\end{aligned}
\tag{22}
$$

where $\tau_i = T_i - T_0$ and $x = a_0$ is the unique solution of

$$
Q_{a_1}^+(x, \tau_1; a_1) = Q_{a_2}^-(x, \tau_2; a_2) \tag{23}
$$

Thus $a_0$ is the time $T_0$ exercise price of the chooser option. It follows that at time $t < T_0$, its price can be expressed as the sum of two second order $Q-$options:

$$
\boxed{V(x, t) = Q_{a_0 a_1}^{++}(x, t_0; t_1; a_1) + Q_{a_0 a_2}^{--}(x, t_0; t_2; a_2)} \tag{24}
$$

where $t_i = t - T_i$ for $i = (0, 1, 2)$.

## 3.3 Partial Barrier Options

A normal down-and-out barrier option is knocked out if the underlying stock price hits a fixed barrier from above at any time during the life of the option. Partial knock-out barrier options (Heynen and Kat 1994) occur in two distinct modes: *start-out* or initial monitoring mode and *end-out* or final monitoring mode.

**Partial-time-start-out-barrier**
Let $(t, T_0)$ denote the barrier monitoring period, where $t$ as usual is the current time. If the option survives to time $T_0$ it converts to a standard European option of strike price $k$ and expiry $T_1 > T_0$. Assume the barrier price is set at $h$. For a down-and-out partial call barrier, the current stock price at time $t$ must be above the barrier. The payoff of this option at $T_0$ is therefore a strike $k$ call option provided the stock price at $T_0$ exceeds the barrier $h$. Let

$$V_h(x, T_0) = Q_k^+(x, \tau; k)\mathbf{1}(x > h) \tag{25}$$

One might be tempted to use the corresponding solution $V_h(x, t)$ as the price of the option. However the barrier option also includes a boundary condition $V_{do}(h, t) = 0$ for all $t < T_0$ that expresses its knock-out when the underlying stock price hits the barrier. This boundary condition can be accounted for by setting $V_{do}(x, t) = V_h(x, t) - \hat{V}_h(x, t)$ where $\hat{V}_h(x, t)$ denotes the *image* of $V(x, t)$ relative the barrier $x = h$ (see Buchen 2001 for details of the method of images for barrier options). In the BS-framework the image solution is given by $\hat{V}(x, t) = (h/x)^\alpha V(h^2/x, t)$ where $\alpha = 2r/\sigma^2 - 1$. Since, for $t < T_0$, $V_h(x, t) = Q_{hk}^{++}(x, t_0, t_1; k)$ the required solution has the representation [1]

$$\boxed{V_{do}(x, t) = Q_{hk}^{++}(x, t_0, t_1; k) - \hat{Q}_{hk}^{-+}(x, t_0, t_1; k)} \tag{26}$$

Once this down-and-out barrier option price is known, corresponding formulas for up-and-out, down-and-in and up-and-in barriers can be obtained immediately as in Buchen (2001).

**Partial-time-end-out-barrier**
The barrier monitoring period is now the interval $(T_0, T_1)$ and it is assumed

[1] In deriving this expression the property: $V(x, t) = f(x, t)\mathbf{1}(x > h)$ implies $\hat{V}(x, t) = \hat{f}(x, t)\mathbf{1}(x < h)$ has been used.

that the option is knocked out whether the underlying stock price hits the barrier from above or below. In this case the time $T_0$ payoff of the option is given by

$$V_{ko}(x,T_0) = V_{do}(x,\tau)\mathbf{1}(x > h) + V_{uo}(x,\tau)\mathbf{1}(x < h) \tag{27}$$

The subscripts here respectively stand for 'knock-out','down-and-out' and 'up-and-out'. Expressions for the standard down-and-out and up-and-out barrier options are given in Buchen (2001) as:

$$\text{Case } h < k : \quad \left\{ \begin{array}{lcl} V_{do}(x,\tau) & = & Q_k^+(x,\tau;k) - \hat{Q}_k^+(x,\tau;k) \\ V_{uo}(x,\tau) & = & 0 \end{array} \right. \tag{28}$$

and

$$\text{Case } h > k : \quad \left\{ \begin{array}{lcl} V_{do}(x,\tau) & = & Q_h^+(x,\tau;k) - \hat{Q}_h^+(x,\tau;k) \\ V_{uo}(x,\tau) & = & (Q_k^+ - \hat{Q}_k^+) - (Q_h^+ - \hat{Q}_h^+) \end{array} \right. \tag{29}$$

Using the image property $\hat{V}(x,t)\mathbf{1}(x > h) = [V(x,t)\mathbf{1}(x < h)]\hat{}$ for any function $V(x,t)$, the following two expressions are obtained for the the end-out partial barrier options:
For $h < k$ and $t < T_0$

$$\boxed{V_{ko}(x,t) = Q_{hk}^{++}(x,t_0,t_1;k) - \hat{Q}_{hk}^{-+}(x,t_0,t_1;k)} \tag{30}$$

while for $h > k$ and $t < T_0$

$$\boxed{V_{ko}(x,t) = Q_{hh}^{++} + Q_{hk}^{-+} - Q_{hh}^{-+} - \hat{Q}_{hh}^{-+} - \hat{Q}_{hk}^{++} + \hat{Q}_{hh}^{++}} \tag{31}$$

The simplifying notation introduced in this paper allows these solutions to be written down in terms of just a handful of second order $Q-$options and their images. This contrasts sharply with the conventional Black-Scholes approach in which the last expression of six $Q-$options will contain twelve rather complicated bivariate normal distribution functions as in Heynen and Kat's (1994) original article. However, it should be stressed that the representations just derived, and all others in this paper, are also valid in a non-BS framework. However, in this case the derivation of analytic expressions for $Q-$options and their images may present insuperable problems.

## 3.4 Cliquet Option

In a cliquet or one-shout option (Thomas 1994), time $T_0$ (the shout time) is a fixed time before expiry $T_1$ at which any payoff can be locked in. For a one-shout call option of strike $k$, the time $T_1$ payoff will be $V(y, T_1) = \max[x-k, y-k, 0]$ where $x, y$ denote the underlying asset price at time $T_0$ and $T_1$ respectively. Thus even if the option finishes out of the money at time $T_1$ ($y < k$), it still pays the amount $(x-k)^+$ that was locked in at time $T_0$. The task is to price the option at time $T_0$ by first considering its payoff at time $T_1$.

If $x < k$, it is fairly clear that

$$V(y, T_1) = (y-k)^+ \qquad \text{and} \qquad V(x, T_0) = Q_k^+(x, \tau; k); \qquad x < k \tag{32}$$

which merely expresses the observation that the option will be a plain vanilla call option of strike price $k$ and expiry $T_1$. On the other hand, if $x > k$ then

$$V(y, T_1) = \max[x-k, y-k] = (x-k) + (y-x)^+ \tag{33}$$

At time $T_1$, $x$ is known and these two terms represent the payoff of $(x-k)$ unit bonds and a strike $x$ call option. It follows that at time $T_0$, in order to avoid arbitrage, the option must have value

$$V(x, T_0) = (x-k)e^{-r\tau} + Q_x^+(x, \tau; x); \qquad x > k \tag{34}$$

The last term here $Q_x^+(x, \tau; x)$ is the price of an at-the-money call option (stock = strike) with time $\tau = T_1 - T_0$ remaining to expiry. In the BS-framework $Q_x^+(x, \tau; x) = xg(\tau)$, where $g(\tau)$ is independent of $x$ and is a function of $\tau$ involving only univariate normals, obtained from Eqs(9) and (10) with $\xi = k = x$.

Putting the two expressions for $V(x, T_0)$ together, there results

$$V(x, T_0) = Q_k^+(x, \tau; k)\mathbf{1}(x<k) + [(x-k)e^{-r\tau} + xg(\tau)]\mathbf{1}(x>k) \tag{35}$$

It is now possible to write down the cliquet option price for any time $t < T_0$ in the form

$$\boxed{V(x, t) = Q_{kk}^{-+}(x, t_0, t_1; k) + e^{-r\tau}Q_k^+(x, t_0; k) + g(\tau)A_k^+(x, t_0)} \tag{36}$$

involving first and second order $Q-$options and a up-asset binary. Alternative but equivalent representations for this cliquet option price also exist.

## 3.5 Extendable Options

An extendable call option (Longstaff 1990) gives its holder the right at time $T_0$ to exercise a standard European call option of strike price $h$; or for a premium $c$, to extend the expiry date to time $T_1 > T_0$ and change the strike from $h$ to $k$. Assume that the parameters are such that $x = a$ and $x = b$ are unique solutions of the equations

$$Q_k^+(a, \tau; k) = c \qquad \text{and} \qquad Q_k^+(b, \tau; k) = b + c - h \tag{37}$$

with $a < h < b$ and $\tau = T_1 - T_0$. A simple graph of the functions involved indicates that this will certainly be the case if $h < ke^{-r\tau} - c$. In these circumstances the option at time $T_0$ will: expire worthless if $x < a$; be extended if $a < x < b$; and be exercised if $x > b$. The $T_0$ payoff of the option can therefore be written in the form

$$V(x, T_0) = [Q_k^+(x, \tau; k) - c][\mathbf{1}(x{>}a) - \mathbf{1}(x{>}b)] + (x - h)\mathbf{1}(x{>}b) \tag{38}$$

Hence at time $t < T_0$ the price of the option is given by

$$\boxed{V(x,t) = \left\{ \begin{array}{l} Q_{ak}^{++}(x, t_0, t_1; k) - Q_{bk}^{++}(x, t_0, t_1; k) + \\ \\ Q_b^+(x, t_0; h - c) - cB_a^+(x, t_0) \end{array} \right.} \tag{39}$$

## 3.6 American Call with Dividend

Consider an American call option of strike price $k$ and expiry date $T_1$ on an underlying asset that pays a single cash dividend of amount $d$ at time $T_0 < T_1$. This problem, first solve by Geske (1979b), while not technically a dual-expiry option, nevertheless does fit into this class – the ex-dividend date $T_0$ playing the role of an expiry date. Assume that $d > k(1 - e^{-r\tau})$ where as usual $\tau = T_1 - T_0$. If this condition is not satisfied it will never be optimal to exercise the American call before time $T_1$. When the condition does hold, it may be optimal to exercise before time $T_1$ and only then, immediately after time $T_0$. Suppose further that just after $T_0$, the stock price drops by the amount of the dividend $d$. Since the stock price process $X_t$ is obviously not continuous it cannot be the solution of a continuous Itô stochastic differential equation. However the process

$$Y_t = \left\{ \begin{array}{ll} X_t - de^{-rt_0}; & t < T_0 \\ X_t; & t > T_0 \end{array} \right. \tag{40}$$

which represents the dividend adjusted stock price process, is continuous at time $T_0$. It is assumed that this process satisfies an Itô stochastic differential equation.

Let $y_0 = Y_{T_0}$. Then just before $T_0$ the option should be continued to be held if $y_0 + d - k < Q_k^+(y_0, \tau; k)$ and should be immediately exercised if $y_0 + d - k > Q_k^+(y_0, \tau; k)$. This follows from the observation that for $T_0 < t < T_1$ the option becomes a standard European option with price $Q_k^+(y, t_1; k)$. Thus the critical $y-$value that determines early exercise is $y = a$, the solution of

$$Q_k^+(y, \tau; k) = y + d - k \tag{41}$$

This solution exists and is unique if the dividend $d$ is sufficiently large to satisfy the inequality stated at the start of this section. It follows that at time $T_0$ the option has value

$$V(y, T_0) = Q_k^+(y, \tau; k)\mathbf{1}(y<a) + (y + d - k)\mathbf{1}(y>a) \tag{42}$$

So its price at time $t < T_0$ can be expressed solely in terms of first and second order $Q-$binaries:

$$\boxed{V(x,t) = Q_{ak}^{-+}(y, t_0, t_1; k) + Q_a^+(y, t_0; k - d); \qquad y = x - de^{-rt_0}} \tag{43}$$

### 3.7 Protection Option

The theoretical framework developed in this paper makes it an easy task to price user defined dual-expiry exotics. For example, consider an option that gives its holder at time $T_0$: a strike $a$, expiry $T_1$ put option if the $T_0$ stock price is below $x = a$; or a strike $b$, expiry $T_1$ call option if the $T_0$ stock price is above $x = b$; and zero otherwise. Both options are in-the-money at time $T_0$. Such an exotic provides a degree of protection if the stock price at $T_0$ has moved outside the channel $a < x < b$. Its price is easily calculated as follows. First obtain the payoff at time $T_0$:

$$V(x, T_0) = Q_a^-(x, \tau; a)\mathbf{1}(x<a) + Q_b^+(x, \tau; b)\mathbf{1}(x>b) \tag{44}$$

and then simply write down the required price at time $t < T_0$ as:

$$\boxed{V(x,t) = Q_{aa}^{--}(x, t_0, t_1; a) + Q_{bb}^{++}(x, t_0, t_1; b)} \tag{45}$$

While this looks superficially very much like the price of a chooser option considered earlier, the difference is substantial.

# 4 Conclusions

A study of the mathematical expressions derived in this paper for dual-expiry exotic options indicates that each contains at least one second order $Q-$option. Additional terms may include other second order $Q-$options, as for example, for chooser, partial barrier, extendable and protection options. These extra terms may also include certain first order binaries. Traditionally, all these option prices have been derived separately by various authors, usually by integrating discounted, time $T_0$ payoffs over particular probability densities. This paper has demonstrated that none of this is really necessary. One need only calculate the prices of certain first and second order binaries, then all dual-expiry options can be priced as linear combinations of these.

There are three decided advantages in using this pricing methodology. The first is that there results a far simpler mathematical representation of the option pricing formulas. The second is that these formulas are valid for a wider class of models of which the Black-Scholes is just the most popular. Finally, the stated expressions show at a glance how these dual expiry exotics may be statically hedged with basic first and second order asset and cash binaries.

Extensions of the method introduced in this paper to multiple-expiry options is relatively straight forward, obviously involving higher order normal distribution functions in the Black-Scholes framework. Examples include ladder options, multiple shout options, Bermudan call options, multiple extendable options and barrier options with discrete monitoring.

# 5 References


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