Investments for the Short and Long Run

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Paul Samuelson: "why we should not make mean log of wealth though years to act are long"

Samuelson (1963, 1969, 1979)

Markowitz (1959, 1976),

Latané (1959), Breiman (1961),

Thorp (1972), Hakansson (1971),

Rubinstein (1976), Cover (1991),

Ziemba & Mulvey (1998),

Browne (1999), Stutzer (2000)

Continuous Time Financial Market

- **primary security accounts**

  S_t^j cum-dividend jth stock

  $j \in \{0, 1, \ldots, d\}$

- **portfolio**

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j \, S_t^j$$

self-financing strategy

$$\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, d_t^d)^\top, \, t \in [0, \infty)\}$$

- **finite expected rate of return**

Assumption 1 *There exists* S^{δ_+} *such that*

$$\frac{1}{\sigma - \tau} E\left(\frac{\frac{S^{\delta}_{\sigma}}{S^{\delta_+}_{\sigma}} - \frac{S^{\delta}_{\tau}}{S^{\delta_+}_{\tau}}}{\frac{S^{\delta}_{\tau}}{S^{\delta_+}_{\tau}}} \,\middle|\, \mathcal{A}_{\tau} \right) \leq K^{\delta_+}_{\tau} < \infty.$$

- **expected growth rate**

$$g^{\delta}_{\tau,\sigma} = \frac{1}{\sigma - \tau} \, E\left(\ln\left(\frac{S^{\delta}_{\sigma}}{S^{\delta}_{\tau}}\right) \,\middle|\, \mathcal{A}_{\tau} \right)$$

$\Longrightarrow$

- **growth optimal portfolio (GOP)** $S^{(\delta_*)}$

$$g^{\delta}_{\tau,\sigma} \leq g^{\delta_*}_{\tau,\sigma}$$

- **nonnegative portfolio** S^{δ}

$\Longrightarrow$

$$E\left(\frac{S^{\delta}_{\sigma}}{S^{\delta_*}_{\sigma}} \,\middle|\, \mathcal{A}_{\tau}\right) \leq \frac{S^{\delta}_{\tau}}{S^{\delta_*}_{\tau}}$$

S^{δ_*} numeraire portfolio

- **excludes** weak form of **arbitrage**:

Under limited liability nothing
can be generated out of nothing !

• **long term growth rate**

$$\tilde{g}^{\delta}_{\infty} \overset{\text{a.s.}}{=} \limsup_{t \to \infty} \frac{1}{t} \ln \left(\frac{S^{\delta}_{t}}{S^{\delta}_{0}} \right)$$

$\Longrightarrow$

$$\tilde{g}^{\delta_*}_{\infty} \geq \tilde{g}^{\delta}_{\infty}$$

Kelly (1956), Thorp (1972),

Karatzas & Shreve (1998), Platen (2004)

If no time limitation $\Longrightarrow$ GOP pathwise best

Continuous Financial Market

- **jth primary security account**

$$dS_t^j = S_t^j \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right)$$

$j \in \{1, 2, \dots, d\}$

- **savings account**

$$S_t^0 = \exp\left\{ \int_0^t r_s \, ds \right\}$$

Assumption 2

Volatility matrix $b_t = [b_t^{j,k}]_{j,k=1}^d$ invertible.

$\Longrightarrow$

- **market price of risk**

$$\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top = b_t^{-1}\,[a_t - r_t\,\mathbf{1}]$$

$$dS_t^j = S_t^j \left(r_t\,dt + \sum_{k=1}^{d} b_t^{j,k}\ \left(\theta_t^k\,dt + dW_t^k\right)\right)$$

- **portfolio**

$$dS_t^\delta = S_t^\delta \left(r_t\, dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j\, b_t^{j,k} \left(\theta_t^k\, dt + dW_t^k\right) \right)$$

jth **fraction**

$$\pi_{\delta,t}^j = \delta_t^j\, \frac{S_t^j}{S_t^\delta}$$

• **logarithm of portfolio**

$$d\ln(S_t^\delta) = g_t^\delta\,dt + \sum_{k=1}^{d}\sum_{j=1}^{d}\pi_{\delta,t}^{j}\,b_t^{j,k}\,dW_t^k$$

• **growth rate**

$$g_t^\delta \overset{\text{a.s.}}{=} \lim_{h\to 0} g_{t,t+h}^\delta \overset{\text{a.s.}}{=} r_t + \sum_{k=1}^{d}\sum_{j=1}^{d}\pi_{\delta,t}^{j}\,b_t^{j,k}\left(\theta_t^k - \frac{1}{2}\sum_{\ell=1}^{d}\pi_{\delta,t}^{\ell}\,b_t^{\ell,k}\right)$$

- **growth optimal portfolio (GOP)** $S_t^{\delta_*}$

growth rate

$$g_t^{\delta_*} \geq g_t^{\delta}$$

fractions

$$\pi_{\delta_*,t} = \left(\pi_{\delta_*,t}^1, \ldots, \pi_{\delta_*,t}^d\right)^\top = \left(b_t^{-1}\right)^\top \theta_t$$

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(\left[r_t + \sum_{k=1}^{d} (\theta_t^k)^2 \right] dt + \sum_{k=1}^{d} \theta_t^k \, dW_t^k \right)$$

- **discounted portfolio** $\bar{S}_t^\delta = \frac{S_t^\delta}{S_t^0}$

$$d\bar{S}_t^\delta = \psi_{\delta,t}^\top \{\theta_t \, dt + dW_t\}$$

- **diffusion coefficient** $\psi_{\delta,t}^\top = (\psi_{\delta,t}^1, \ldots, \psi_{\delta,t}^d) = \bar{S}_t^\delta \, \pi_{\delta,t}^\top \, b_t$

- **drift** $\alpha_t^\delta = \psi_{\delta,t}^\top \, \theta_t$

- **aggregate diffusion coefficient**

$$\gamma_t^\delta = \sqrt{\psi_{\delta,t}^\top \, \psi_{\delta,t}}$$

- **optimal portfolio**

$\bar{S}^{\tilde{\delta}}$ **optimal** if for all $\bar{S}^{\delta}$ with

$$\gamma_t^{\delta} = \gamma_t^{\tilde{\delta}}, \qquad \alpha_t^{\delta} \leq \alpha_t^{\tilde{\delta}}$$

generalization of mean-variance optimality

Markowitz (1952, 1959)

Assumption 3

$$|\theta_t| = \sqrt{\theta_t^{\top}\, \theta_t}, \quad \pi^0_{\delta_*,t} \neq 1$$

not risk neutral market, GOP risky

- **two fund separation**

Tobin (1958), Sharpe (1964), Platen (2002, 2005a)

optimal portfolio

$$d\bar{S}_t^\delta = \bar{S}_t^\delta \, (J_t^\delta)^{-1} \; \theta_t^\top \, (\theta_t \, dt + dW_t)$$

$$\pi_{\delta,t} = \left(\pi_{\delta,t}^1, \ldots, \pi_{\delta,t}^d\right)^\top = (J_t^\delta)^{-1} \, \pi_{\delta_*,t}$$

- **risk aversion coefficient**

$$J_t^\delta = \frac{1 - \pi^0_{\delta_*,t}}{1 - \pi^0_{\delta,t}}$$

Optimal portfolio has fraction in GOP

and remainder in savings account.

- for **optimal portfolio** S^δ

  aggregate volatility

$$|b_\delta(t)| = (J_t^\delta)^{-1} |\theta_t|$$

  risk premium

$$p_\delta(t) = \frac{\alpha_t^\delta}{\bar{S}_t^\delta} = (J_t^\delta)^{-1} \; |\theta_t|^2 = |b_\delta(t)| \, |\theta_t|$$

$\Longrightarrow$ capital market line

$\Longrightarrow$ Markowitz efficient frontier

$$a_\delta(t) = r_t + p_\delta(t) = r_t + |b_\delta(t)| \, |\theta_t| = r_t + \sqrt{|b_\delta(t)|^2} \, |\theta_t|$$

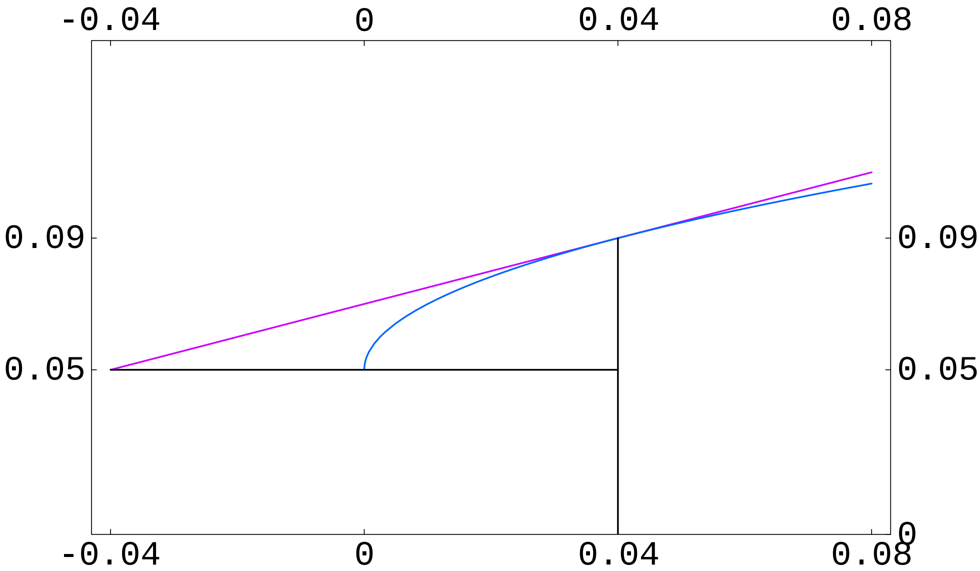

Figure 1: Efficient frontier.

- **Sharpe ratio**

  for general portfolio S^δ

  Sharpe (1964)

$$s_t^\delta = \frac{p_\delta(t)}{b_\delta(t)} \leq |\theta_t|$$

  Sharpe ratio maximizers

  form optimal portfolios !

- **growth rate**

  for optimal portfolio S^δ

$$g_\delta(t) = r_t + \sqrt{|b_\delta(t)|^2}\,|\theta_t| - \frac{1}{2}\,|b_\delta(t)|^2$$

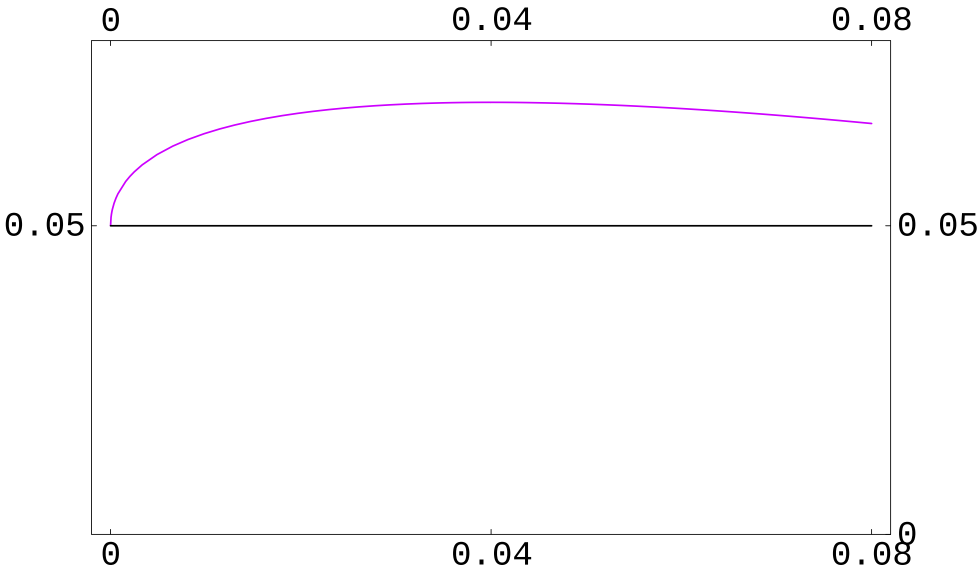

Figure 2: Growth rates of optimal portfolios.

- **risk premium**

  for general portfolio S^δ

$$p_\delta(t) = \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t}\, b^{j,k}_t\, \theta^k_t$$

- **systematic risk parameter**

  $S^{\delta_{\rm opt}}$ risky optimal portfolio

$$\beta_\delta(t) = \frac{\frac{d}{dt}\langle \ln(S^\delta), \ln(S^{\delta_{\rm opt}})\rangle_t}{\frac{d}{dt}\langle \ln(S^{\delta_{\rm opt}}), \ln(S^{\delta_{\rm opt}})\rangle_t}$$

  $\langle X, Y\rangle_t$ covariation

$\Longrightarrow$ **risk premium**

$$p_\delta(t) = \beta_\delta(t)\, p_{\delta_{\rm opt}}(t)$$

capital asset pricing model (CAPM) if

$$S^{\delta_{\rm opt}} = S^{\delta_+}$$

Sharpe (1964), Lintner (1965),

Mossin (1966) and Merton (1973)

Assumption 4 *Each investor forms an optimal portfolio.*

$\Longrightarrow$ **Market portfolio is optimal portfolio.**

$\Longrightarrow$ **CAPM**

assumes no equilibrium,

no expected utility, no Markovianity

Assumption 5 *Fundamental relationships are invariant under changes of currency denomination.*

$\Longrightarrow$

$$S_t^{\delta_+} = S_t^{\delta_*}$$

$\Longrightarrow$ **Market portfolio equals GOP.**

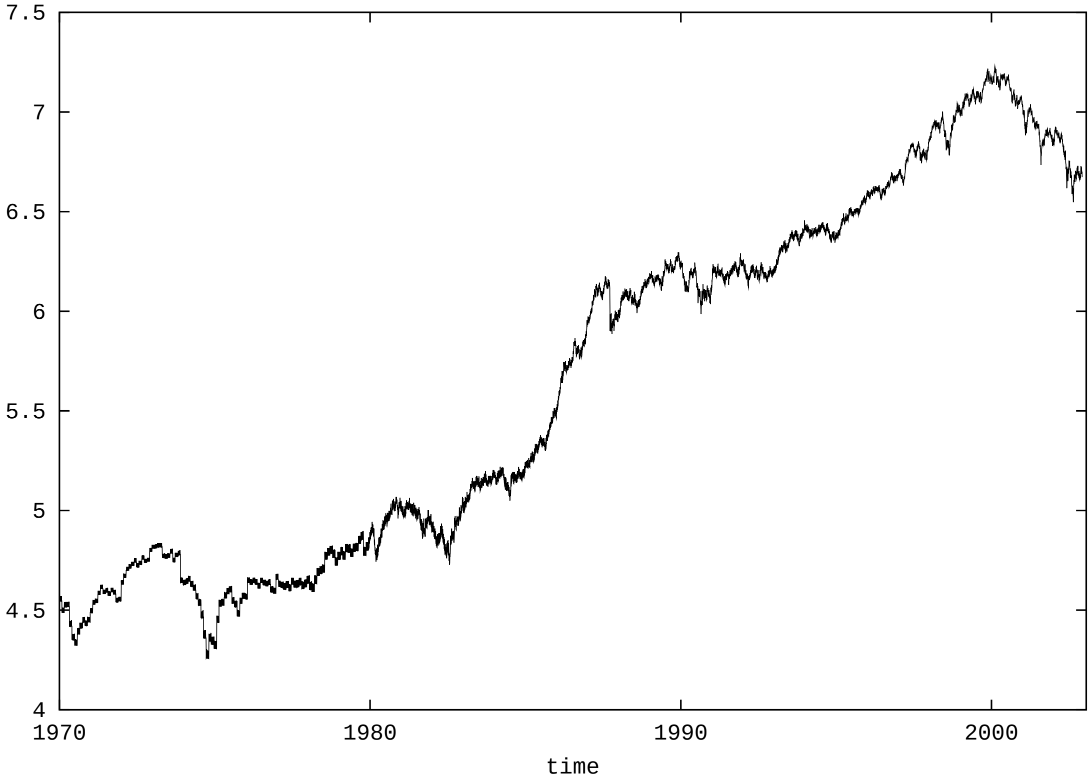

Figure 3: Logarithm of discounted MSCI.

- **discounted GOP drift**

$$\alpha_t^{\delta_*} = \bar{S}_t^{\delta_*} \, |\theta_t|^2$$

- **discounted GOP**

$$d\bar{S}_t^{\delta_*} = \alpha_t^{\delta_*} \, dt + \sqrt{\alpha_t^{\delta_*} \, \bar{S}_t^{\delta_*}} \, d\tilde{W}_t$$

Squared Bessel process of dimension four !

• **underlying value as transformed time**

$$\varphi(t) = \varphi(0) + \int_0^t \alpha_s^{\delta_*}\, ds$$

$$d\sqrt{\bar{S}_t^{\delta_*}} = \frac{3\,\alpha_t^{\delta_*}}{8\sqrt{\bar{S}_t^{\delta_*}}}\, dt + \frac{1}{2}\sqrt{\alpha_t^{\delta_*}}\, d\tilde{W}_t$$

$\Longrightarrow$

$$\varphi(t) - \varphi(0) = 4\left\langle \sqrt{\bar{S}^{\delta_*}} \right\rangle_t$$

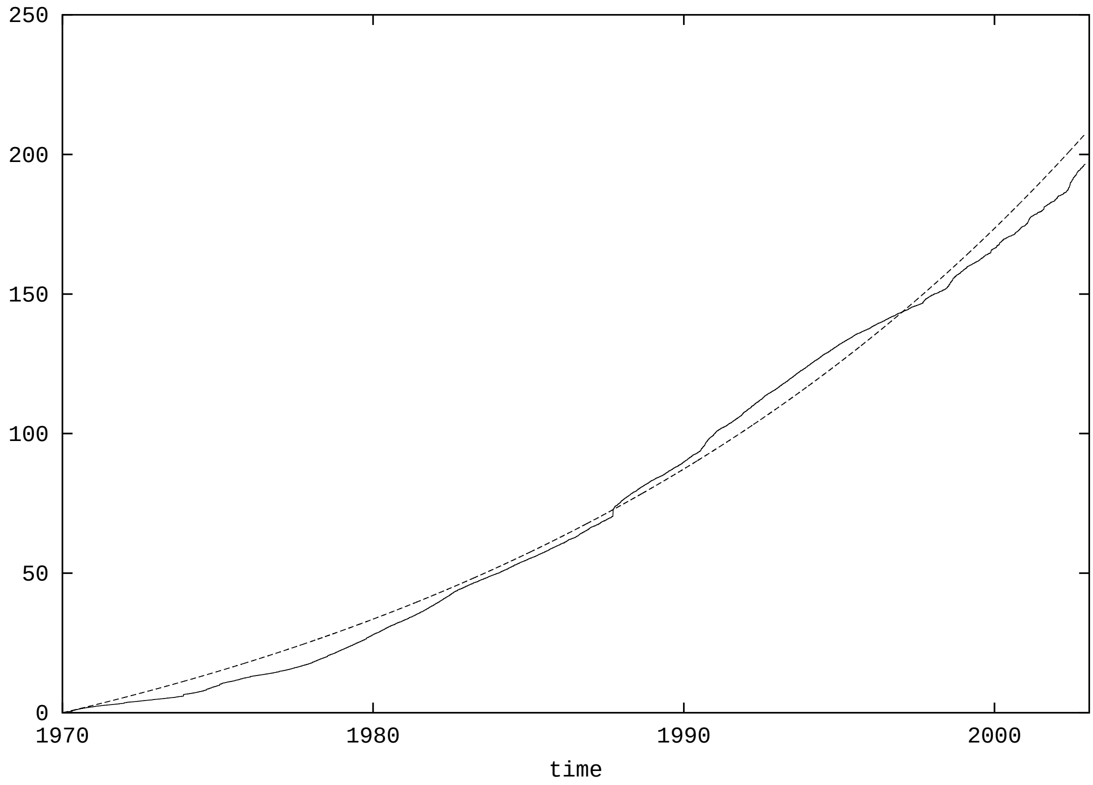

Figure 4: Increments of φ-time of discounted MSCI and theoretical transformed time.

Assumption 6

Discounted GOP drift is an exponential function.

$$\alpha_t^{\delta_*} = \alpha_0 \exp\{\eta\, t\}$$

η - **net growth rate**

$\Longrightarrow$

- **minimal market model (MMM)**

  Platen (2002, 2005b)

- **volatility of the GOP (market portfolio)**

$$|\theta_t| = \sqrt{\frac{\alpha_t^{\delta_*}}{\bar{S}_t^{\delta_*}}}$$

$\Longrightarrow$ **Student *t* distributed log-returns with four degrees of freedom**

Markowitz & Usmen (1996),

Fergusson & Platen (2005)

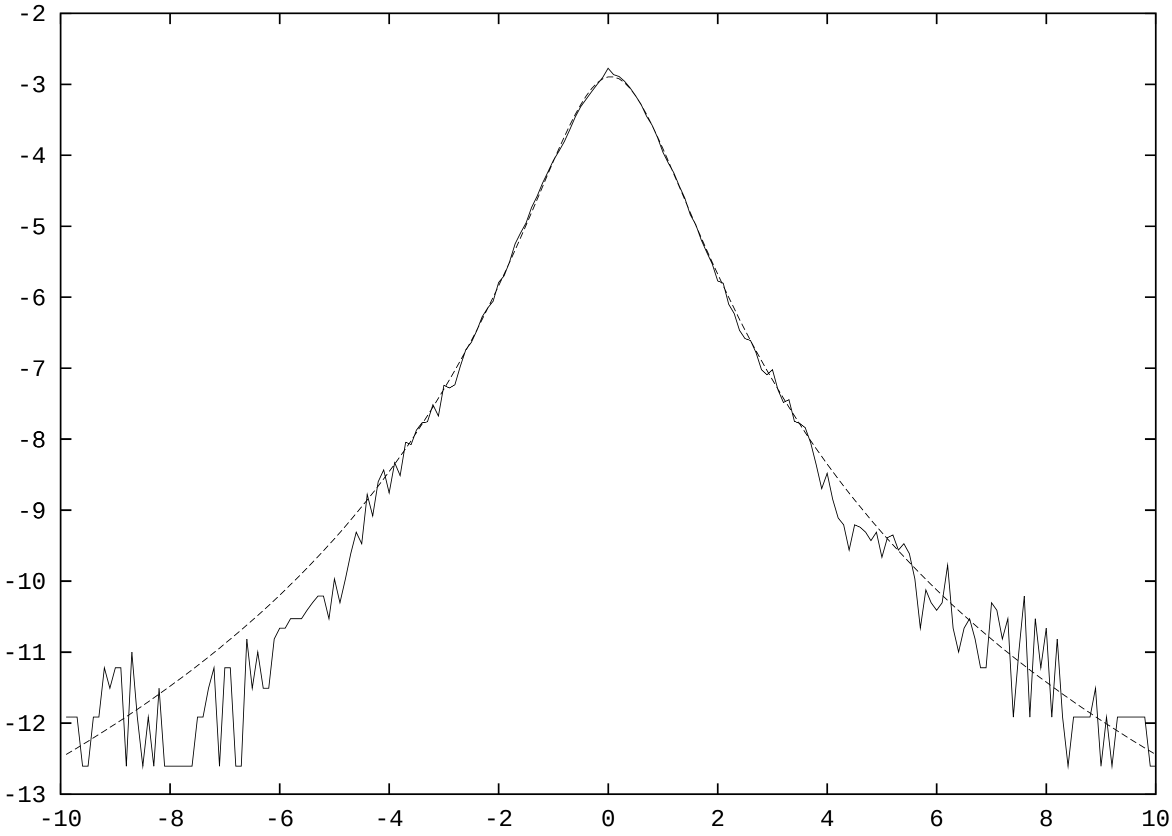

Figure 5: Log-histogram of WSI log-returns.

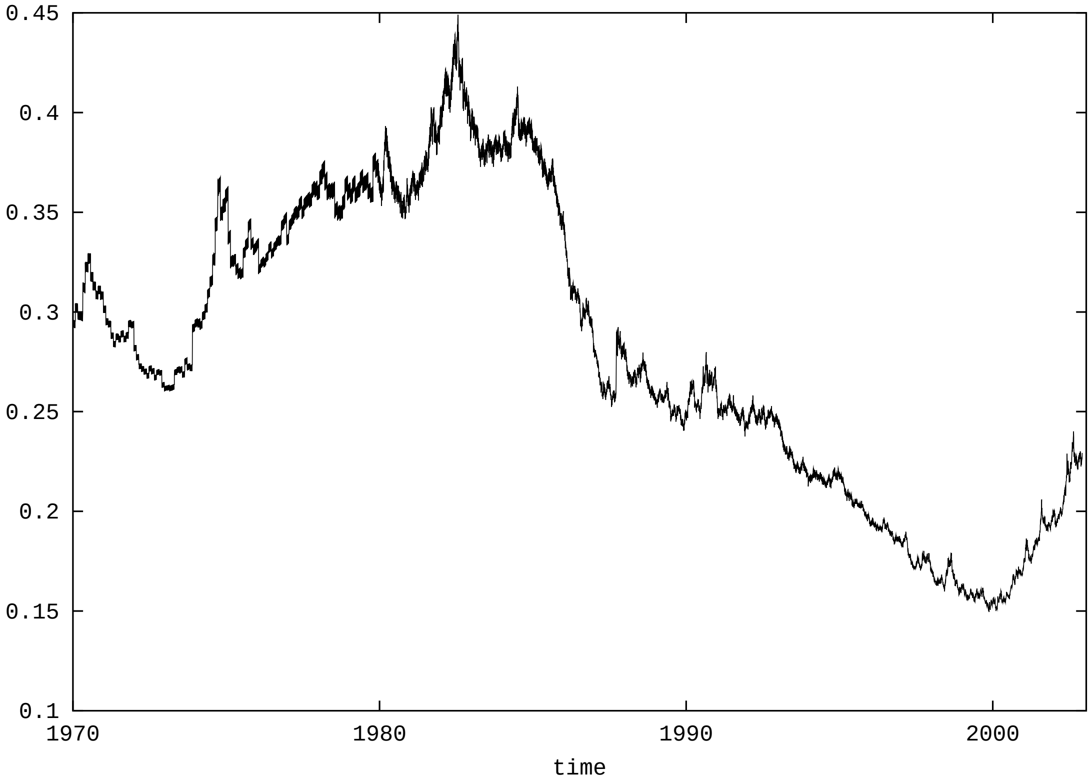

Figure 6: Volatility of the MSCI under the MMM.

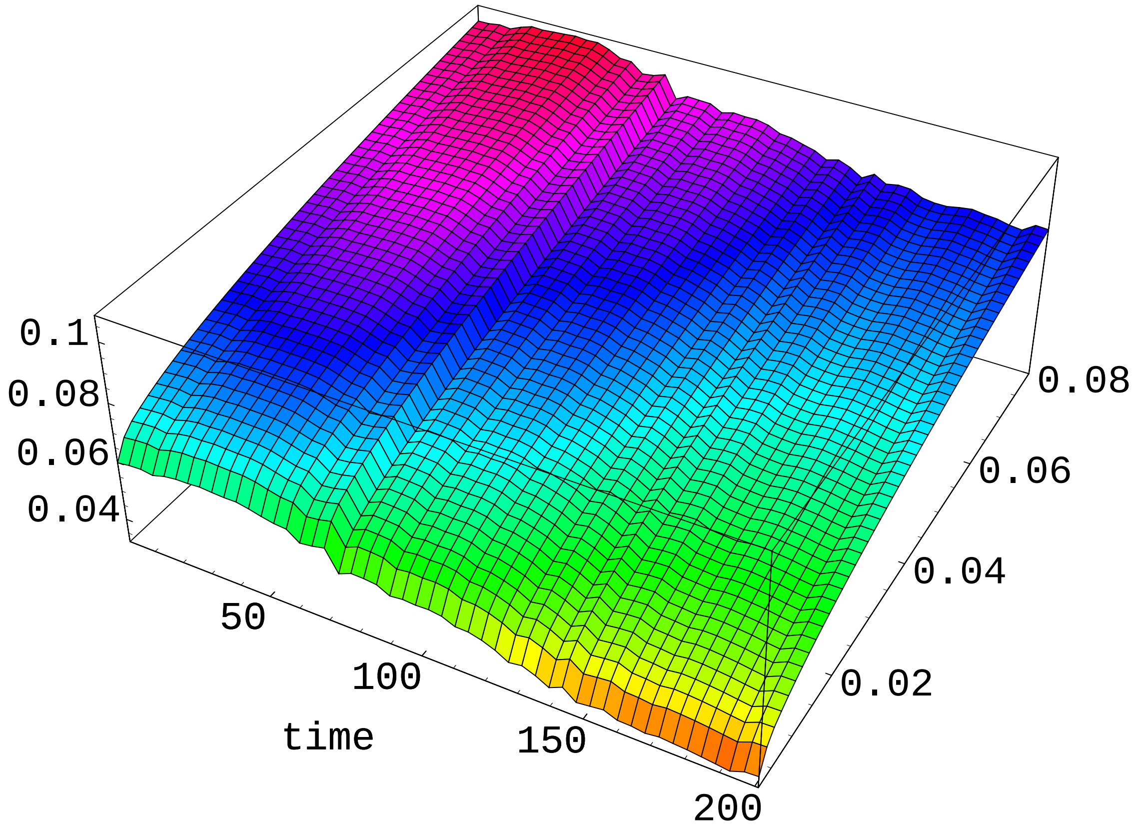

Figure 7: Efficient frontier over time.

Utility Maximization

- **utility function**

$$U : [0, \infty) \to \Re$$

assume derivative U' can be inverted

$$E\left(U\left(\bar{S}_T^\delta\right)\right) \to \max$$

maximize over strictly positive, discounted fair portfolios $\bar{S}^\delta$

Platen (2005b) $\quad \Longrightarrow$

$\Longrightarrow$ **Expected utility maximizing portfolio is optimal portfolio.**

- **risk aversion coefficient**

$$J_t^\delta = \frac{1}{1 - \frac{\hat{S}_t^0}{\hat{u}(t,\hat{S}_t^0)} \frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}_t^0}}$$

$$\hat{u}(t, \hat{S}_t^0) = E\left(U'^{-1}(\lambda\, \hat{S}_T^0)\, \hat{S}_T^0 \,\middle|\, \mathcal{A}_t\right)$$

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