Some New "Stochastic Calculus" Look on the Technical Analysis of the Stock Prices

Albert N. Shiryaev

Steklov Mathematical Institute and Moscow Lomonosov State University

(Sydney, 12 August 2005)

The main motivation of the talk is based on idea to obtain a mathematical explanation of some practical methods ("when to buy, when to sell", *etc.*) of the Technical Analysis which have as usual only a descriptive character.

As is well known, the "fundamentalists" are trying to explain

WHY the stock price moves;

they make their decisions by looking at the state of the "economy at large"; they define a stock value and calculate proper stock prices in view of its estimated future values; they build their analysis upon the assumption that the actions of market operators are "rational".

As to the “technicians” they concentrate on the local peculiarities of the markets, they emphasize “mass behavior”, “market moods”; they start their analysis from an idea that stock price movement is “the product of supply and demand”; their basic concept is the following: the analysis of past stock prices helps us to see future prices because past prices take future prices into account; they try to explain

HOW the stock prices move.

CONTENTS

§ 1. "Mathematization" of the "Kagi charts" and "Renko charts" from the Technical Analysis

Let $X = (X_t)_{t \geq 0}$ be a stock price.

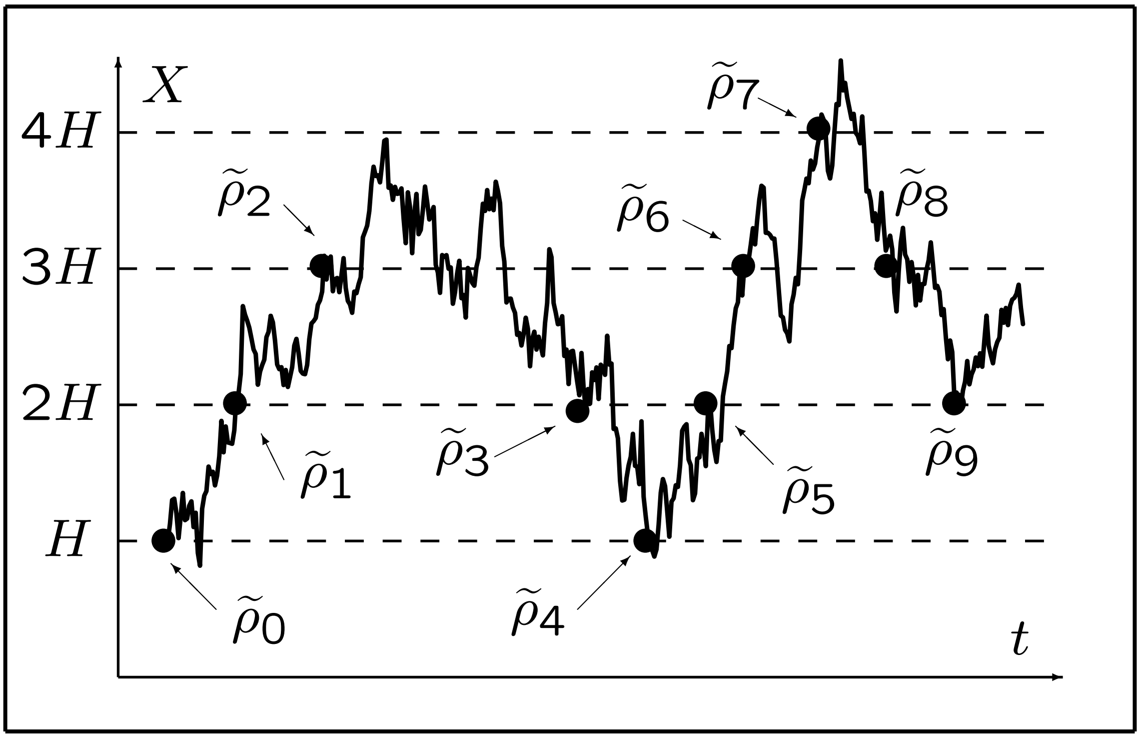
The Japanese "Kagi chart" and "Renko chart" (also called the price ranges) give methods to forecast price *trends* from price changes which exceed either a certain *range* H or a certain rate H. The price range or rate H is determined in advance. (In Japan, popular price ranges are ¥5, 10, 20, 50, 100, 200.) Greater price ranges are used for stocks with higher prices because their upward and downward movements are larger.

$$R \to |X|$$
$$K \to \max X - X$$

<u>Renko-construction</u>:

Step I: We construct $(\widetilde{\rho}_i)$:

$$\widetilde{\rho}_0 = 0,$$
$$\widetilde{\rho}_{n+1} = \inf \left\{ t > \widetilde{\rho}_n : |X_t - X_{\widetilde{\rho}_n}| = H \right\}, \quad n \geq 1.$$

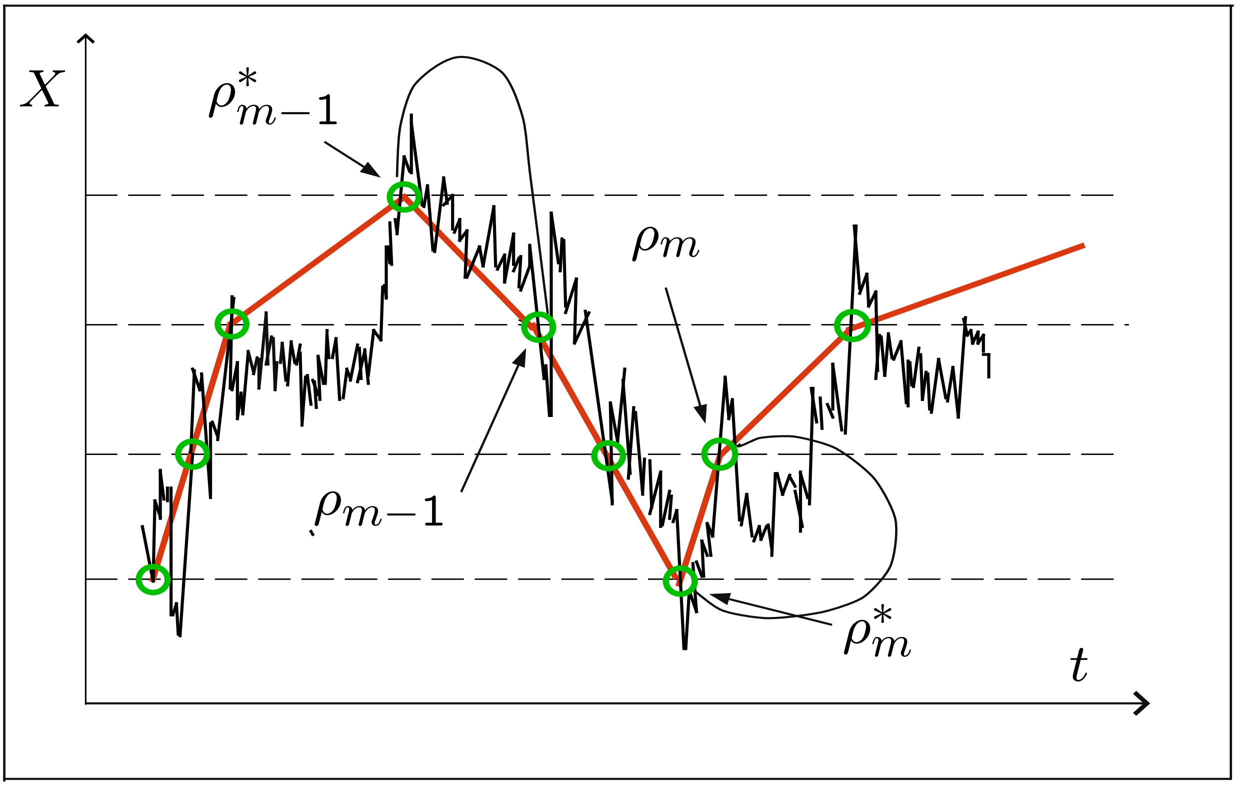


Step II: Construction

$$(\widetilde{\rho}_n) \longrightarrow (\rho_m, \rho_m^*).$$

We look at all $\widetilde{\rho}_n$ such that

$$\left(X_{\widetilde{\rho}_n} - X_{\widetilde{\rho}_{n-1}}\right)\left(X_{\widetilde{\rho}_{n-1}} - X_{\widetilde{\rho}_{n-2}}\right) < 0.$$



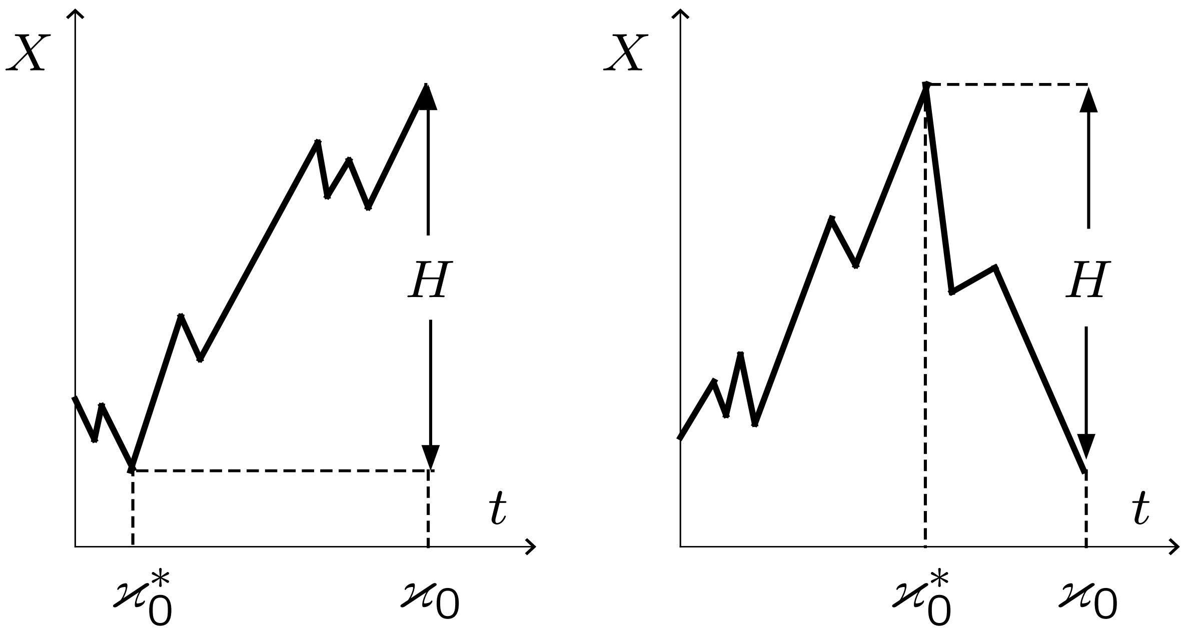
ρ_m is a Markov time

ρ_m^* is a non-Markov time

Kagi construction:

$$\varkappa_0 = \inf\left\{u > 0 : \max_{[0,u]} X - \min_{[0,u]} X = H\right\}$$

$$\varkappa_0^* = \begin{cases} \inf\left\{u \in [0,\varkappa_0] : X_u = \min\limits_{[0,\varkappa_0]} X\right\} \\ \qquad \text{if } X_{\varkappa_0} = \max\limits_{[0,\varkappa_0]} X \\ \inf\left\{u \in [0,\varkappa_0] : X_u = \max\limits_{[0,\varkappa_0]} X\right\} \\ \qquad \text{if } X_{\varkappa_0} = \min\limits_{[0,\varkappa_0]} X \end{cases}$$



Next step: we define by induction

$$\varkappa_{n+1} = \begin{cases} \inf\left\{u > \varkappa_n : \max\limits_{[x_n,u]} X - X_u = H\right\} \\ \qquad \text{if } X_{\varkappa_n} - X_{\varkappa_n^*} = H \\ \\ \inf\left\{u > \varkappa_n : \max\limits_{[x_n,u]} X - X_u = H\right\} \\ \qquad \text{if } X_{\varkappa_n} - X_{\varkappa_n^*} = -H \end{cases}$$

$$\varkappa^*_{n+1} = \begin{cases} \inf\left\{u \in [\varkappa_n, \varkappa_{n+1}] : X_u = \max\limits_{[\varkappa_n,\varkappa_{n+1}]} X\right\} \\ \qquad \text{if } X_{\varkappa_n} - X_{\varkappa_n^*} = H \\ \\ \inf\left\{u \in [\varkappa_n, \varkappa_{n+1}] : X_u = \min\limits_{[\varkappa_n,\varkappa_{n+1}]} X\right\} \\ \qquad \text{if } X_{\varkappa_n} - X_{\varkappa_n^*} = -H \end{cases}$$

Kagi and Renko variation (on $[0, T]$):

$$K_T(X; H) = \sum_{n=1}^{N} |X_{\varkappa_n^*} - X_{\varkappa_{n-1}^*}|, \qquad N = N_T(X; H),$$

$$R_T(X; H) = \sum_{n=1}^{M} |X_{\rho_n^*} - X_{\rho_{n-1}^*}|, \qquad M = M_T(X; H).$$

Kagi and Renko volatilities (on $[0,T]$):

$$k_T(X;H) = \frac{K_T(X;H)}{M_T(X;H)},$$
$$r_T(X;H) = \frac{R_T(X;H)}{M_T(X;H)}.$$

Theorem. *If $X = \sigma B$ then*

1) $k_T(\sigma B; H) \sim 2H$, $\quad N_T \sim \dfrac{T\sigma^2}{H^2}$ (P-*a.s.*), *and*

$$K_T = k_T N_T \overset{\mathsf{P}}{\sim} \frac{2T\sigma^2}{H^2};$$

2) $r_T(\sigma B; H) \overset{\mathsf{P}}{\sim} 2H$, $\quad M_T \sim \dfrac{T\sigma^2}{2H^2}$ (P-*a.s.*), *and*

$$R_T = r_T M_T \overset{\mathsf{P}}{\sim} \frac{T\sigma^2}{H}.$$

Results of the statistical analysis of some stock prices

$X = (X_t)_{t \geq 0}$ — Future on Index SP500 (Emini-SP500 Futures) 1 point = $\$\,50$

2002-2003 (471 trading days)

$\Delta = 1$ sec

X_t is the value of the last transaction at time t.

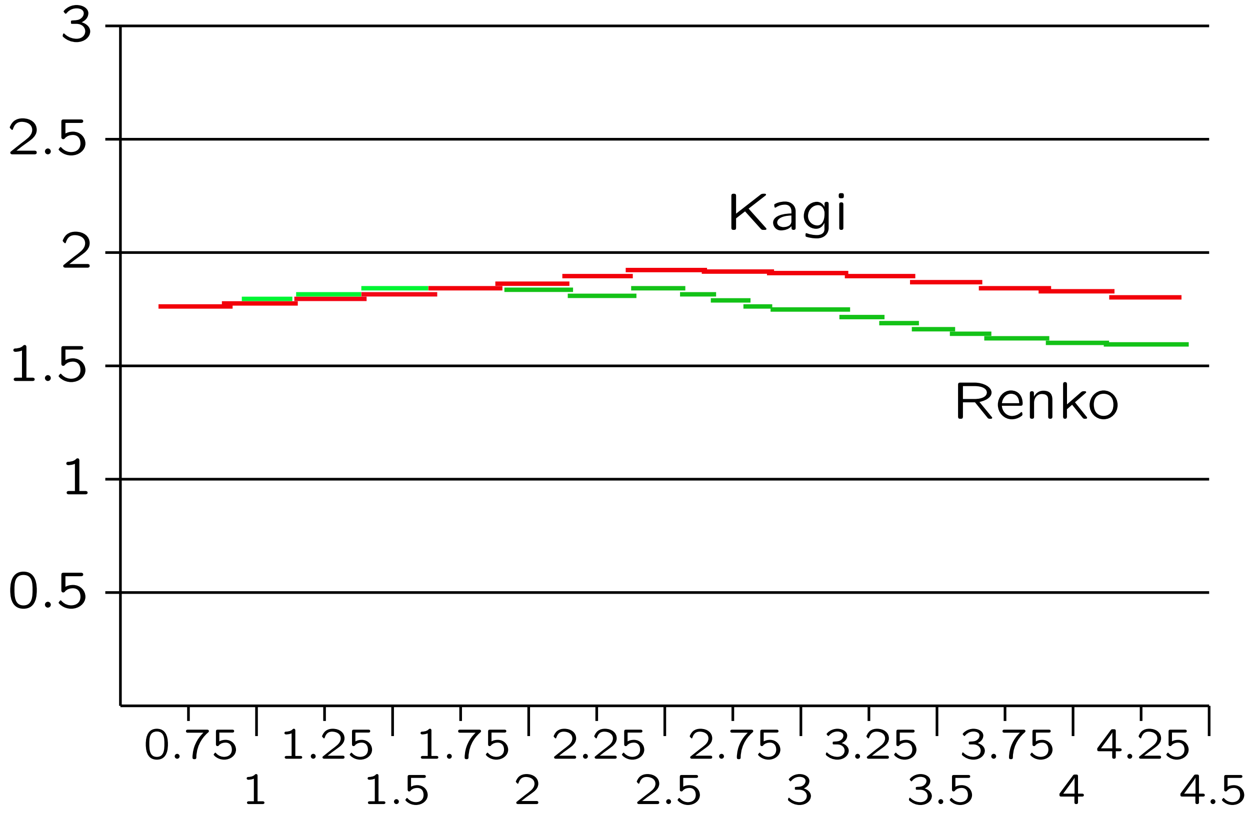
RENKO

H	1	1.25	1.5	2	2.25	2.5	3	4
$\frac{r_T(X;H)}{H}$	1.83	1.84	1.86	1.88	1.86	1.88	1.80	1.69

KAGI

H	1	1.25	1.5	2	2.25	2.5	3	4
$\frac{k_T(X;H)}{H}$	1.83	1.85	1.85	1.89	1.91	1.93	1.92	1.87

Almost the same results are valid for Futures on Index Nasdaq 100 (Emini-Nasdag100 Futures), 1 point = $\$\,20$



For EESR (United Energy System of Russia)

$$\frac{r_T(X;H)}{H} \sim 1.99 \sim \frac{k_T(X;H)}{H}.$$

Let us say that X-market has

$r(H)$-property if $\mathsf{E} r_T(X;H) \sim r(H) \cdot H$

$k(H)$-property if $\mathsf{E} k_T(X;H) \sim k(H) \cdot H$

(For a Brownian motion $r(H) = k(H) = 2$.)

Define Renko strategy $\gamma^R = (\gamma^R_t)_{t\geq 0}$ with

$$\gamma^R_t = \sum_{n\geq 1} \mathsf{sgn}\left(X_{\rho_{n-1}} - X_{\rho^*_{n-1}}\right) I_{[\rho_{n-1},\rho^*_{n-1})}(t) \\ \times \left(I(k(H)\geq 2) - I(k(H)<2)\right), \quad t \geq 0,$$

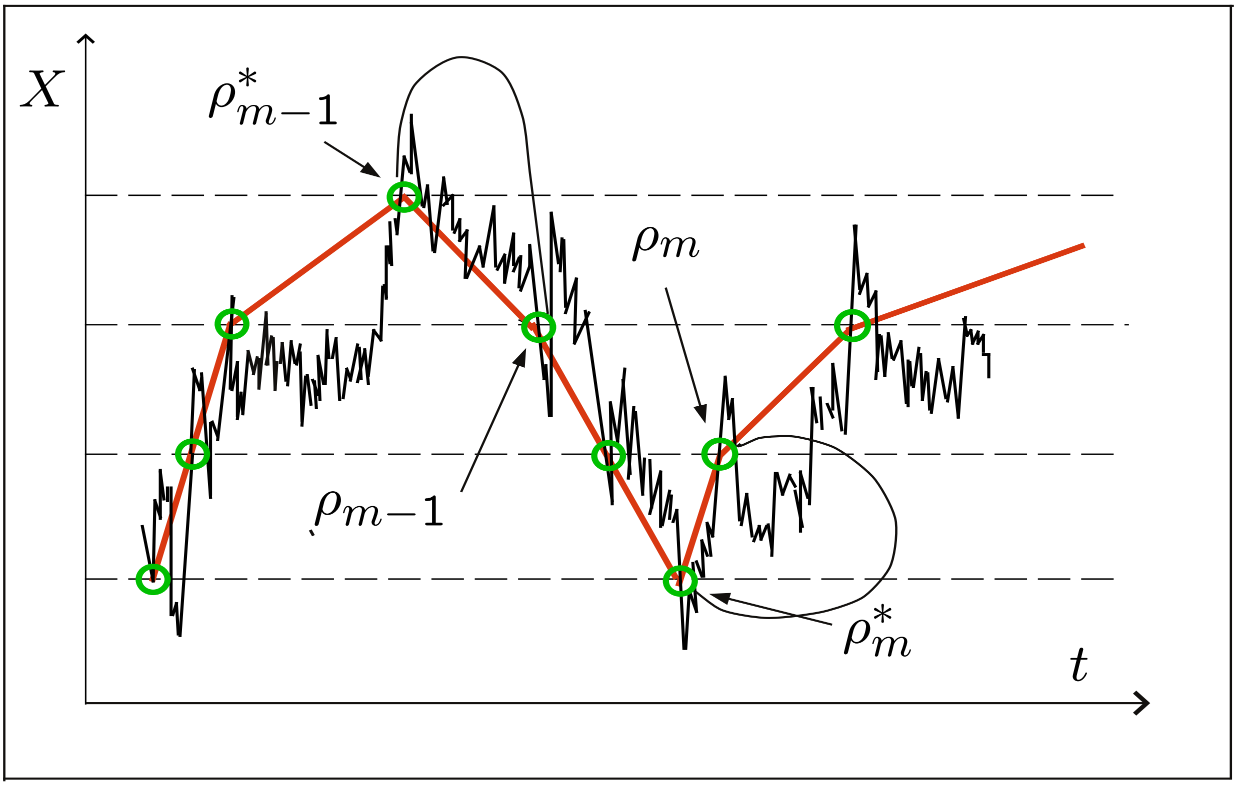
and the corresponding capital

$$C^{\gamma^R}_t = \int_0^t \gamma^R_u \, dX_u - \lambda \int_0^t |d\gamma^R_u|.$$

Then

$$\lim_{t\to\infty} \mathsf{E}\frac{C^{\gamma^R}_t}{M_t} = |r(H) - 2| \cdot H - 2\lambda.$$

The similar result is valid for the Kagi strategy $\gamma^K = (\gamma^K_t)_{t\geq 0}$.



If $R(H) > 2$	we buy in times	$\rho_{m-2}, \rho_m, \ldots$
	we sell in times	$\rho_{m-1}, \rho_{m+1}, \ldots$
	$(\uparrow\uparrow)$ $(\downarrow\downarrow)$	

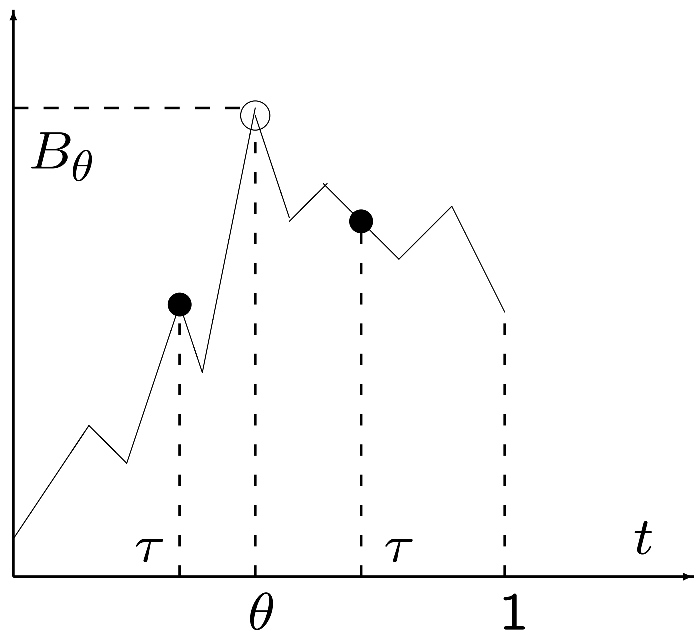
If $R(H) < 2$	we buy in times	$\rho_{m-1}, \rho_{m+1}, \ldots$
	we sell in times	$\rho_{m-2}, \rho_m, \ldots$
	$(\uparrow\downarrow)$ $(\downarrow\uparrow)$	

§ 2. Prediction of time of maximum value of prices observable on time interval $[0, T]$

I would like to present now several our probability and statistical approaches to solving some other problems of the technical analysis.

Problem. *When to sell stock optimally?*

We shall describe prices by a Brownian motion $B = (B_t)_{0 \le t \le 1}$; θ is a point of maximum of B: $B_\theta = \max\limits_{0 \le t \le 1} B_t$.



Suppose that we begin to observe this process at time $t = 0$ ("morning time"), and, using only past observations, we stop at time τ declaring "alarm" about selling. It is very natural to try to solve the following problem: to find "optimal" times τ^* and τ^{**} such that either

$$\inf_{0 \le \tau \le 1} \mathsf{E}|B_\theta - B_\tau|^2 = \mathsf{E}|B_\theta - B_{\tau^*}|^2$$

or

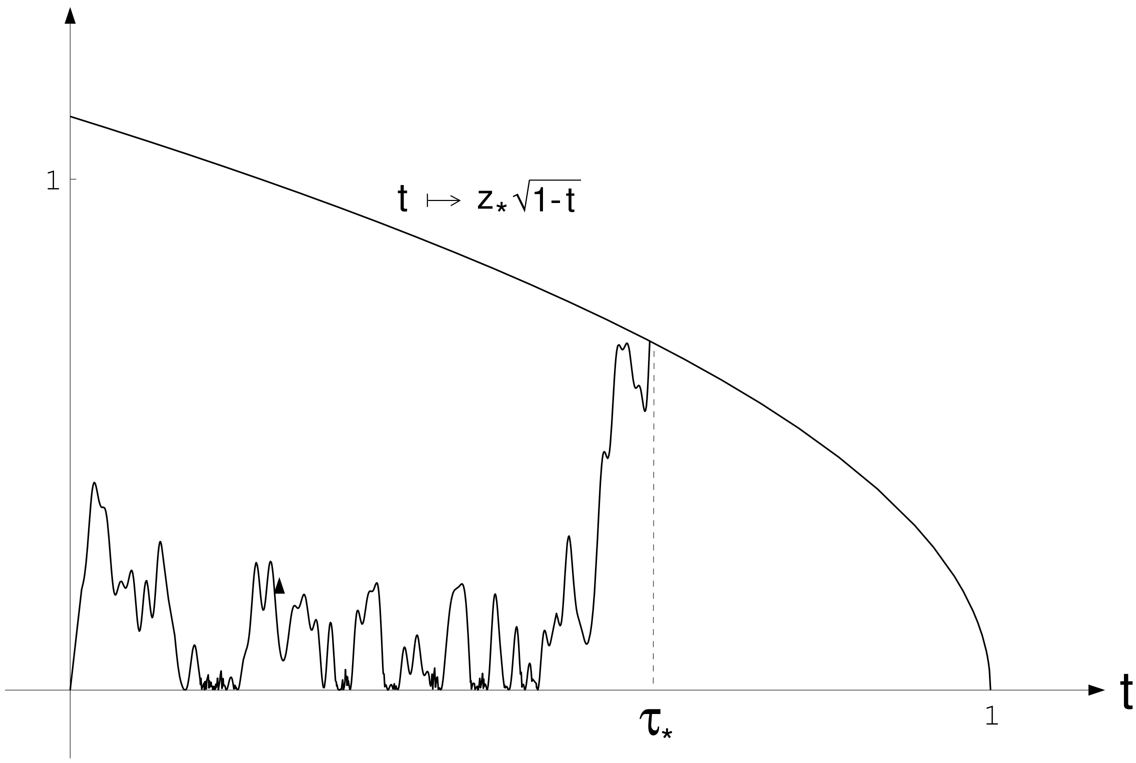
$$\inf_{0 \le \tau \le 1} \mathsf{E}|\theta - \tau| = \mathsf{E}|\theta - \tau^{**}|.$$

For us it was a little bit surprising that here the optimal stopping times coincide: $\tau^{**} = \tau^*$. The solution shows that

$$\tau^* = \inf\left\{t \le 1 : \max_{s \le t} B_s - B_t \ge z_* \sqrt{1-t}\right\},$$

where z_* is a certain (known) constant ($z_* = 1.12\ldots$).

This problem belongs to the theory of optimal stopping and method of its solution is based on reducing to the special *free-boundary* problem.



It is interesting to note that

$$\mathsf{E}\tau^* = 0.55\ldots, \qquad \mathsf{D}\tau^* = 0.05\ldots.$$

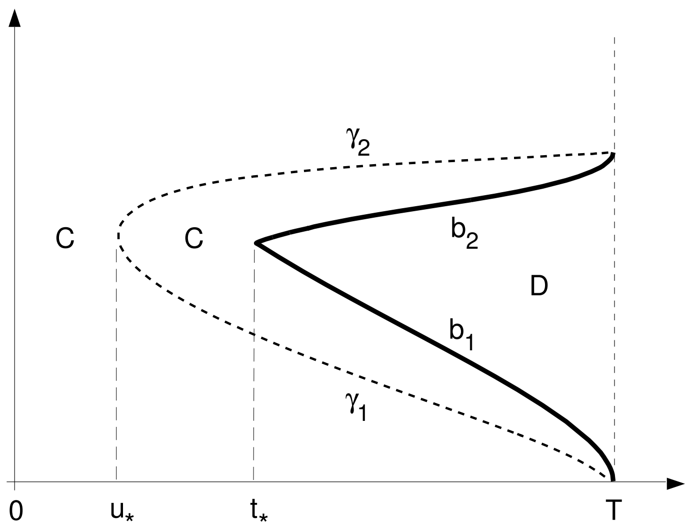
The cases $B_t^\mu = \mu t + B_t$ instead of B_t are more complicated. If $\mu > 0$ and μ is away from 0, then

$$\tau^* = \inf\{t \le 1 : b_1(t) \le S_t^\mu - B_t^\mu \le b_2(t)\}$$

where

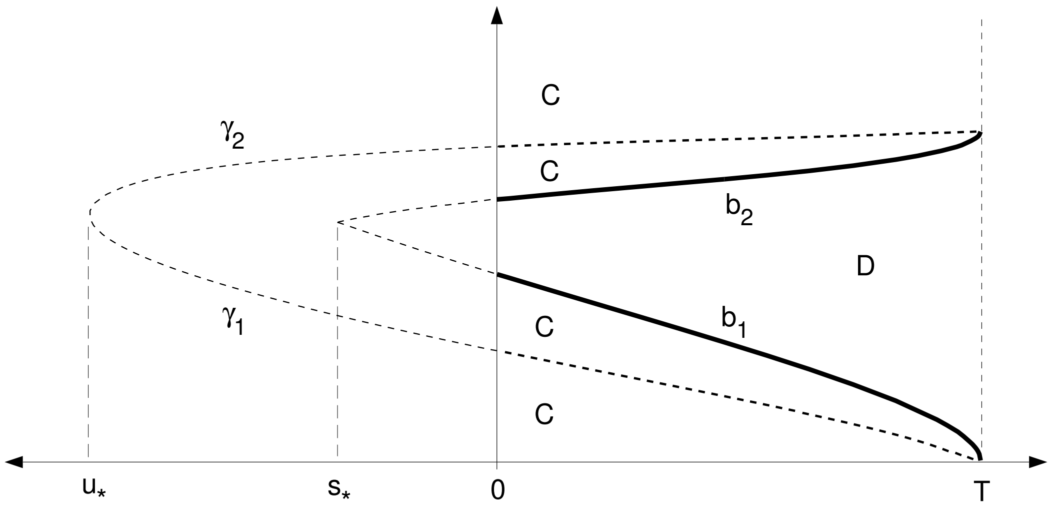
$$B_t^\mu = \mu t + B_t, \qquad S_t^\mu = \max_{u \le t} B_u^\mu,$$

and $b_1(t)$ and $b_2(t)$ have the following form:

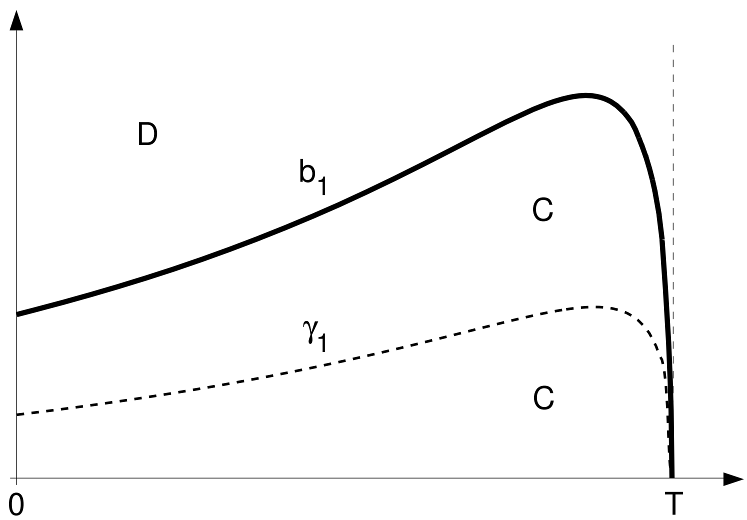


⟨Here C is the area of continuation of observations, D is the stopping area.⟩

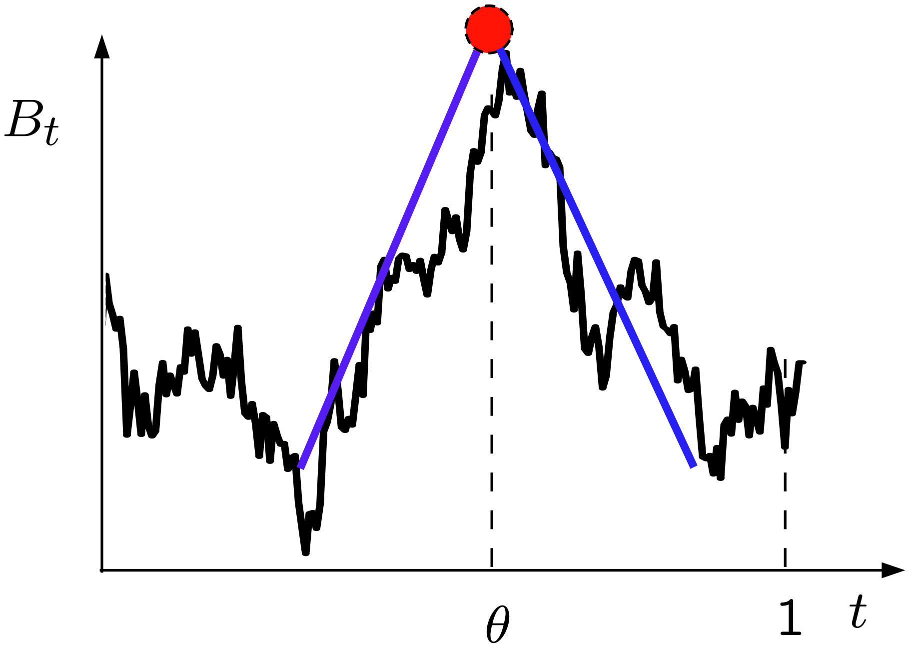
If $\mu > 0$ and μ is close to 0, then the corresponding picture has the following form:



For $\mu < 0$ and if μ is far from 0, the picture is as follows:



In the considered problem, the time θ is a "change point" of the changing of the directions of trend



Solution of the problem

$$\text{“}\inf_{\tau} \mathsf{E}|B_\tau - B_\theta|^2\text{”}$$

or the problem "$\inf_\tau \mathsf{E}|\tau-\theta|$" depends, of course, on the construction at any time t a "good" prediction of the change point θ. The natural estimate of θ should be based on the *a posteriori* probability

$$\pi_t = \mathsf{P}(\theta \le t \,|\, \mathcal{F}_t^B),$$

where $\mathcal{F}_t^B = \sigma(B_s, s \le t)$.

Stochastic analysis shows that

$$\pi_t = 2\varphi\left(\frac{S_t - B_t}{\sqrt{1-t}}\right) - 1, \qquad S_t = \max_{u \le t} B_u,$$

that explains appearing of the expression

$$\frac{S_t - B_t}{\sqrt{1-t}}$$

which is involved above in the definition of optimal stopping time

$$\tau^* = \inf\left\{t \le 1 : \frac{S_t - B_t}{\sqrt{1-t}} \ge z_*\right\}.$$

Statistics $S_t - B_t$ is appearing in many problems of the financial mathematics and financial engineering (and, generally, in the mathematical statistics under name *CUSUM statistics*).

Now we are going to tackle the following problem, which is interesting, e.g., from the point of view of the quickest detection of arbitrage.

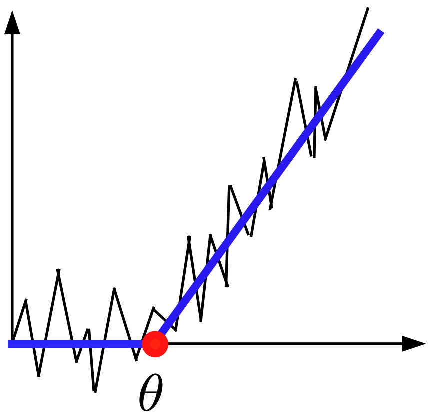
§ 3. Quickest detection of time of appearing of arbitrage

Problem. Suppose we observe the prices

$$X_t = r(t-\theta)^+ + \sigma B_t$$

or

$$dX_t = \begin{cases} \sigma\, dB_t, & t \le \theta, \\ r\, dt + \sigma\, dB_t, & t > \theta. \end{cases}$$

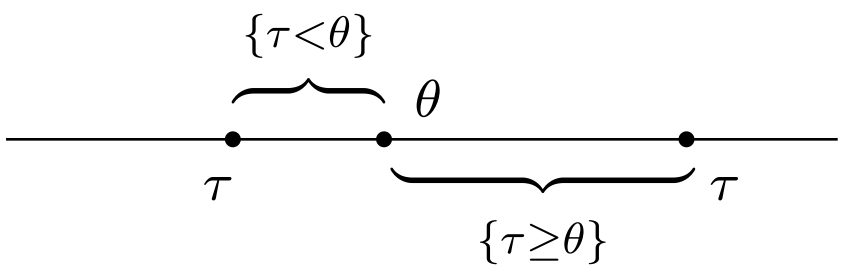


Here a "change point" θ is considered as a time of appearing of arbitrage. (Brownian motion's prices correspond to the *non-arbitrage* situation. Brownian motion with drift corresponds to a case of *arbitrage*.)

One very difficult question here is "what is θ?". There are two approaches. In the first one we assume that θ is a *random variable*. Suppose that τ is time of "alarm" θ. Consider two events

$$\{\tau < \theta\} \quad \text{and} \quad \{\tau \ge \theta\}.$$

The set $\{\tau < \theta\}$ is the event of a false alarm with a (false alarm) probability $\mathsf{P}(\tau < \theta)$.



From a financial point of view an interesting characteristic of the event $\{\tau \geq \theta\}$ is a delay time $\mathsf{E}(\tau - \theta \,|\, \tau \geq \theta)$ or $\mathsf{E}(\tau - \theta)^+$. These considerations lead to the following problem: in the class $\mathfrak{M}_\alpha = \{\tau : \mathsf{P}(\tau < \theta) \leq \alpha\}$, i.e., in the class of stopping times with the probability of false alarm $\mathsf{P}(\tau < \theta)$ which less or equal the fixed level α, one need to find optimal stopping $\tau_\alpha^* \in \mathfrak{M}_\alpha$ such that

$$\inf_{\tau \in \mathfrak{M}_\alpha} \mathsf{E}(\tau - \theta)^+ = \mathsf{E}(\tau_\alpha^* - \theta)^+.$$

It turned out that it is not a simple problem if we consider an arbitrary distribution for θ. However, there exists one case when we may solve this problem in implicit form. This case is the following.

Assume that θ has the *exponential* distribution:

$$\mathsf{P}(\theta = 0) = \pi \quad \text{and} \quad \mathsf{P}(\theta > t \,|\, \theta > 0) = e^{-\lambda t},$$

where λ is a given positive constant and $\pi \in [0,1)$. This assumption is very reasonable. Indeed, for $A < a < b < B$

$$\lim_{\lambda \to 0} \mathsf{P}\Big(\theta \in (a,b) \mid \theta \in (A,B)\Big) = \frac{|b-a|}{|B-A|}.$$

It means that in limit ($\lambda \to 0$) the conditional distribution of θ is *uniform*, that is, in some sense the *worst* possible case from point of view of uncertainty of time of appearing of a change point θ.

We describe now the results about structure of the optimal stopping time τ_α^*.

Denote

$$\pi_t = \mathsf{P}(\theta \le t \,|\, \mathcal{F}_t^X),$$

where $\mathcal{F}_t^X = \sigma(X_s, s \le t)$.

This process satisfies the following nonlinear stochastic differential equation:

$$d\pi_t = \Big(\lambda - \frac{r^2}{\sigma^2}\,\pi_t^2\Big)(1-\pi_t)\,dt + \frac{r}{\sigma^2}\,\pi_t(1-\pi_t)\,dX_\tau$$

with $\pi_0 = \pi$.

Then it turns out that an optimal stopping time τ_α^* has the following form:

$$\tau_\alpha^* = \inf\{t : \pi_t \ge B_\alpha^*\},$$

where (for case $\pi = 0$, for simplicity)

$$B_\alpha^* = 1 - \alpha.$$

Second formulation of the quickest detection of arbitrage assumes that θ is simply a *parameter* from $[0,\infty)$. In this case we denote by P_θ the distribution of the process X under the assumption that a change point is occurred at time θ.

By P_∞ we denote the distribution of X under assumption that there is no change point at all.

Denote for given $T > 0$

$$\mathfrak{M}_T = \{\tau : \mathsf{E}_\infty \tau \le T\}$$

the class of stopping time for which the mean time $\mathsf{E}_\infty \tau$ before (false) alarm is less or equal to T.

Put also

$$\mathbb{C}(T) = \inf_{\tau \in \mathfrak{M}_T} \sup_\theta \operatorname{ess\,sup}_\omega \mathsf{E}_\theta \Big[(\tau - \theta)^+ \,|\, \mathcal{F}_\theta \Big](\omega).$$

We proved that for each $T > 0$ in the class $\mathfrak{M}_T$ there exists an optimal strategy with the following structure: declare alarm at time

$$\tau_T^* = \inf \Big\{ t : \max_{u \le t} X_u - X_t \ge a^*(T) \Big\},$$

where $a^*(T)$ is a certain constant. It is interesting to note that (if $r^2/(2\sigma^2) = 1$)

$$\mathbb{C}(T) \sim \log T, \qquad T \to \infty.$$

The given method, based on the

"CUSUM statistics $\max X - X$",

also is *asymptotically* optimal for more tractable criteria

$$\mathbb{D}(T) = \inf_{\tau \in \mathfrak{M}_T} \sup_{\theta} \mathsf{E}_\theta(\tau - \theta \,|\, \tau \geq \theta)$$

(We don't know what is an optimal method for $\mathbb{D}(T)$-criterion.) Asymptotically, again

$$\mathbb{D}(T) \sim \log T, \qquad T \to \infty.$$

§ 4. Drawdowns, downfalls

From given above exposition we observe importance of the "$\max X - X$"-characteristics for taking optimal decisions. Now I would like to discuss that statistics and related ones in the problems of measure of *risk*. There is a special terminology for such an object which related to words "drawdown", or "downfall".

In practice a "drawdown" on time interval $[0, t]$ is defined as the percent change in a manager's net asset value

- from any newly established peak to a subsequent through,
- from a high "water mark" to the next low "water mark".

From the theoretical point of view,

> *Drawdown* is a statistical measure of risk for investments; a competitor to the standard measure of risk such as return probability, VaR, Sharpe ration, etc.

There are many different definitions of drawdown's characteristics, which measure the decline in net asset value from the historic high point.

In one financial paper I read that

> ... Measuring risk through extreme losses is a very appealing idea. This is indeed how financial companies perceive risks. This explains the popularity of loss statistics as the maximum drawdown and maximum loss...

and

> ... it does not seem possible to derive exact results for the expected maximum drawdown.

Looking forward:

- What kinds of drawdowns should we expect over any given investment horizon?
- How many drawdowns should be experienced?
- How big?

Under the

Commodity Futures Trading Commission's
(CFTC)

mandatory disclosure regime managed futures advisors are obliged to disclose, as part of their capsule performance records, their

"*worst peak-to-valley drawdown*".

We shall demonstrate here some our theoretical calculations related to drawdowns.

Let $B_t^\mu = \mu t + \sigma W_t$ be a Brownian motion with drift, $W_0 = 0$.

There are several interesting characteristics related to

Range,

Drawdowns,

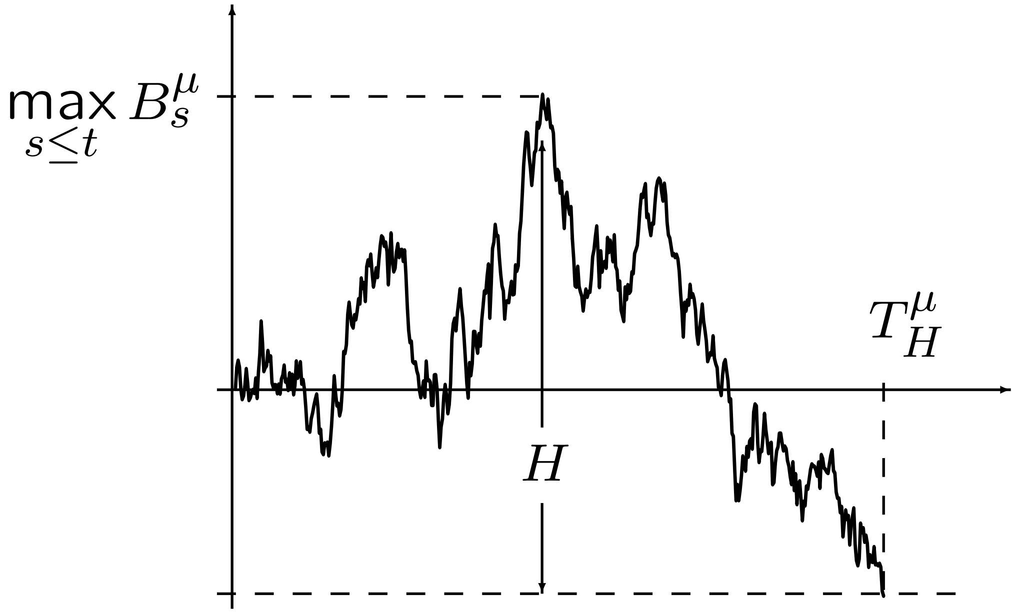
Downfalls,. . .

Range:

$$\boxed{R_t^\mu = \max_{s \le t} B_s^\mu - \min_{s \le t} B_s^\mu}$$

Statistics T_H^μ for B^μ:

$$\boxed{T_H^\mu = \inf \left\{ t \ge 0 : \max_{s \le t} B_s^\mu - B_t^\mu \ge H \right\}}$$



If $\mu = 0$:

$$\mathsf{E}T_H^0 = \left(\frac{H}{\sigma}\right)^2, \qquad \mathsf{E}\max_{t\le T_H^0} B_t^0 = H,$$

$$\mathsf{D}T_H^0 = \frac{2}{3}\left(\frac{H}{\sigma}\right)^4, \qquad \mathsf{E}e^{-\lambda T_H^0} = \frac{1}{\cosh\left(\frac{H}{\sigma}\sqrt{2\lambda}\right)}.$$

If $\mu \neq 0$:

$$\mathsf{E}T_H^\mu = \frac{\sigma^2}{2\mu^2}\left[\exp\left\{\frac{2\mu}{\sigma^2}H\right\} - 1 - \frac{2\mu}{\sigma^2}H\right],$$

$$\mathsf{E}\max_{t\le T_H^\mu} B_t^\mu = \frac{\sigma^2}{2\mu}\left[\exp\left\{\frac{2\mu}{\sigma^2}H\right\} - 1\right].$$

Towards a problem from Kolmogorov's diary (1944):

...For free (or not) random walk: How X_t drops when X_t falls for the first time (on $(t-\Delta, t)$) from above to some level ξ? To all appearance, certainly very steeply!..

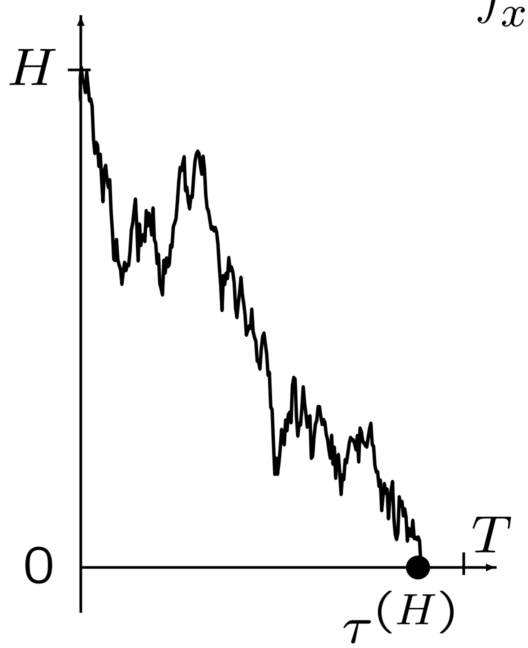
$$B_t^{(H)} = H + B_t, \quad B_0^{(H)} = H, \quad B_t = B_t^0$$

$$\tau^{(H)} = \inf\{u : B_u^{(H)} = 0\}$$

$$F(t) = \mathsf{P}\Big(\tau^{(H)} \le t \mid \min_{s \le T} B_s \le 0\Big)$$

$$f(t) = \frac{dF(t)}{dt} = \frac{H\sqrt{T}}{2G(H/\sqrt{T})}\, t^{-3/2} e^{-H^2/2t}, \quad t \le T$$

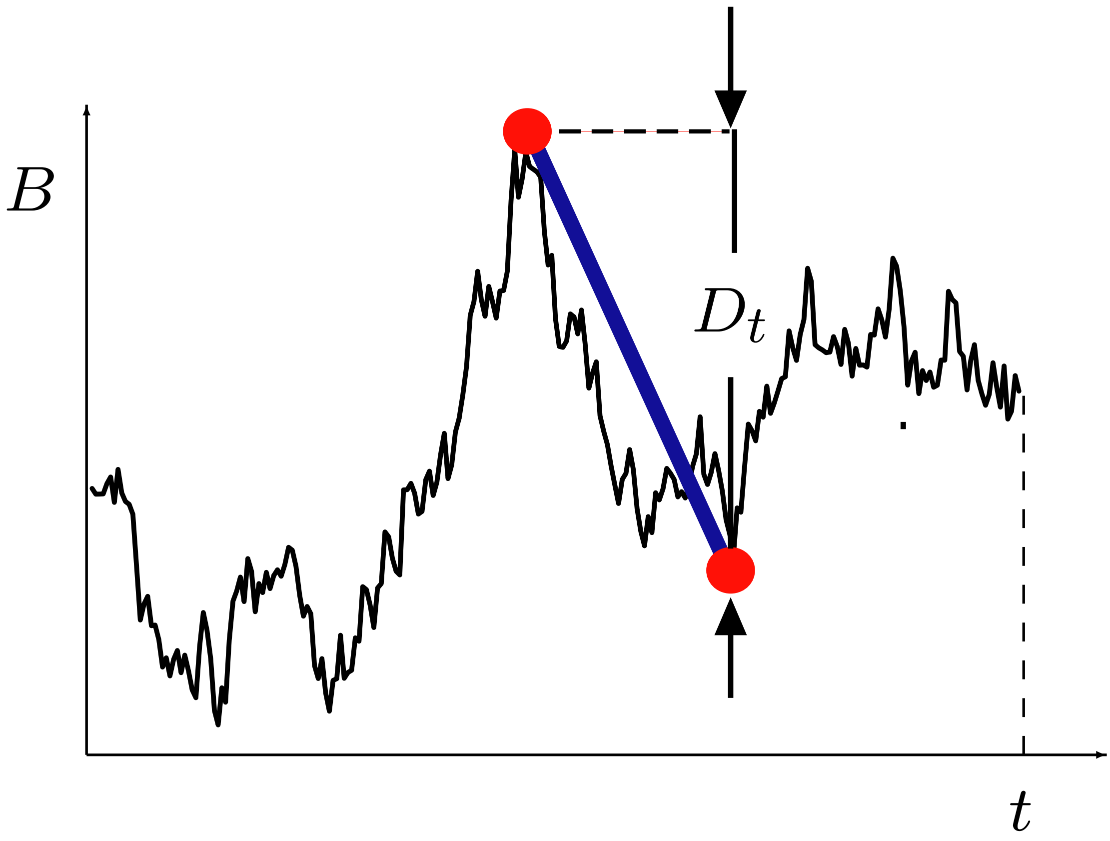
$$G(x) = \int_x^\infty e^{-u^2/2}\, du$$



The following three characteristics of drawdowns are the most important:

Maximum drawdown $\longrightarrow$ $\boxed{D_t = \max_{0 \le s \le s' \le t} (B_s - B_{s'})}$

(cf. $R_t = \max_{0 \le s, s' \le t}(B_s - B_{s'})$; so $D_t \le R_t$).



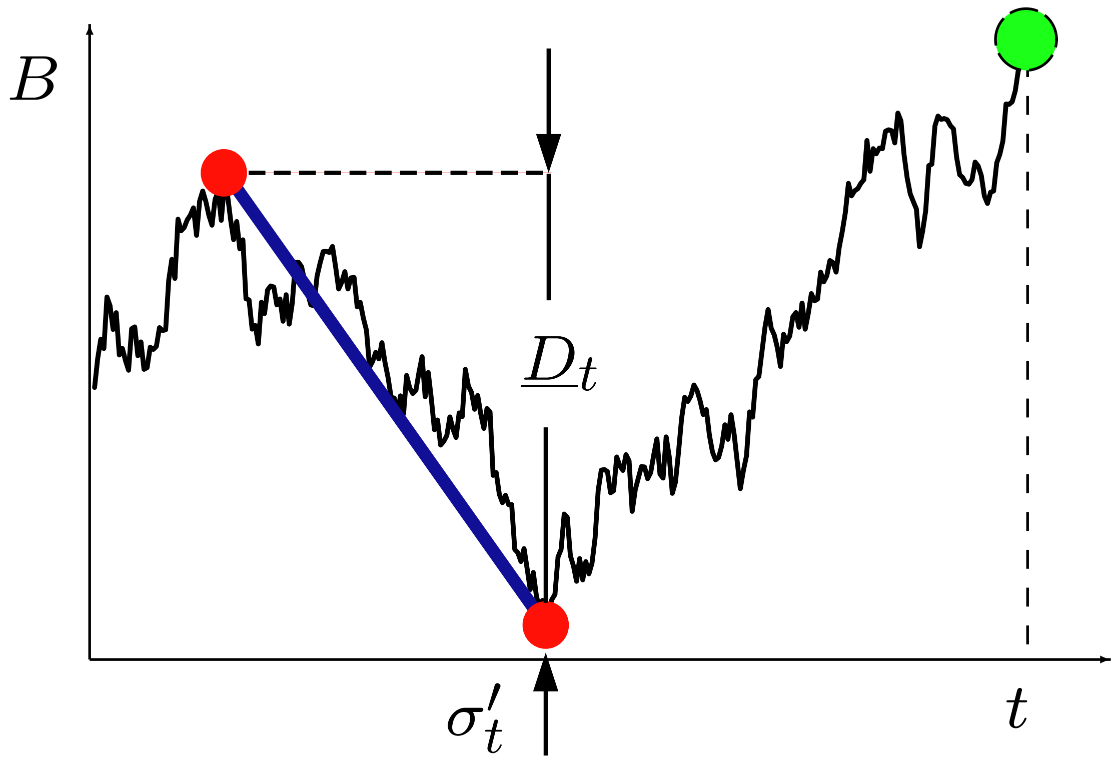
Drawdown from high "water mark" to the next low "water mark" $\longrightarrow$

$$\boxed{\overline{D}_t = B_{\sigma_t} - \min_{\sigma_t \le s' \le t} B_{s'}}$$
$$= \max_{0 \le s \le t} B_s - \min_{\sigma_t \le s' \le t} B_{s'}$$

(where $\sigma_t = \inf\{s \le t : B_s = \max_{u \le t} B_u\}$.

Drawdown from previous high "water mark" to the lowest "water mark" $\longrightarrow$ $\boxed{\underline{D}_t = \max_{0 \leq s \leq \sigma'_t} B_s - B_{\sigma'_t}}$

(where $\sigma'_t = \inf\{s \leq t : B_s = \min_{u \leq t} B_u\}$.



General results on D, $\overline{D}$, $\underline{D}$ for B:

$$(1)\quad \overline{D}_t = \underline{D}_t$$

$$(2)\quad D_t = \max(\overline{D}_t, \underline{D}_t)$$

$$(3)\quad D_t \overset{\text{law}}{=} \max_{s \le t} |B_s|$$

$$(4)\quad \overline{D}_t \overset{\text{law}}{=} \max_{g_t \le s \le t} |B_s|$$

where $g_t = \sup\{s \le t : B_s = 0\}$.

Distributional results on D_1, $\overline{D}_1$ for a standard Brownian motion $B = B^\circ$:

$$(5)\quad \mathsf{P}(D_1 \le x) = \mathsf{P}\Big(\max_{s \le 1} |B_s| \le x\Big) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\Big\{ - \frac{\pi^2 (2n+1)^2}{8x^2} \Big\}$$

$$\boxed{\mathsf{E} D_1 = \mathsf{E} \max_{s \le 1} |B_s| = \sqrt{\tfrac{\pi}{2}} = 1.2533\ldots}$$

$$\mathsf{E} D_t = \sigma \sqrt{t} \sqrt{\tfrac{\pi}{2}} \quad (\text{for } \sigma B^\circ \text{ on } [0,t])$$

(6) $$\mathsf{P}(\overline{D}_1 \le x) = \mathsf{P}\Big(\max_{g_1 \le s \le 1} |B_s| \le x\Big) = F_{\overline{D}_1}(x)$$

$$f_{\overline{D}_1}(x) = \frac{dF_{\overline{D}_1}(x)}{dx} = \sqrt{\frac{8}{\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k e^{-\frac{1}{2}k^2 x^2}$$

$$\boxed{\mathsf{E}\overline{D}_1 = \sqrt{\frac{8}{\pi}} \log 2 = 1.1061}$$

Note that

$$f_{R_1}(x) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-k^2 x^2/2}, \quad x > 0,$$

$$F_{K_1}(x) = \mathsf{P}\Big(\max_{s \le 1} |b_s| \le x \Big),$$

$$f_{K_1}(x) = \frac{dF_{K_1}(x)}{dx} = 8x \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-k^2 x^2},$$

where $b = (b_s)_{s \le 1}$ is a Brownian bridge ($b_s = B_s - sB_1$).

We note that

$$f_{R_1}(x) = \sqrt{\tfrac{2}{\pi}}\, \tfrac{1}{x}\, f_{K_1}(x).$$

So

$$\mathsf{E} R_1 = \sqrt{\tfrac{8}{\pi}} = 1.5957\ldots$$

and

$$\begin{array}{ccccc} \mathsf{E}\overline{D}_1 & \le & \mathsf{E}D_1 & \le & \mathsf{E}R_1 \\ \sqrt{\dfrac{8}{\pi}} \log 2 & \le & \sqrt{\dfrac{\pi}{2}} & \le & \sqrt{\dfrac{8}{\pi}} \\ 1.1061\ldots & \le & 1.2533\ldots & \le & 1.5957\ldots \end{array}$$

Lemma.

(1) $\overline{\mathbb{D}}_t \stackrel{\text{law}}{=} \underline{\mathbb{D}}_t$

(2) $\mathbb{D}_t = \begin{cases} \overline{\mathbb{D}}_t = \underline{\mathbb{D}}_t & \textit{on } \{\sigma_t \le \sigma'_t\} \\ \max(\overline{\mathbb{D}}_t, \underline{\mathbb{D}}_t) & \textit{on } \{\sigma_t > \sigma'_t\} \end{cases}$

(3) $\max(\overline{\mathbb{D}}_t, \underline{\mathbb{D}}_t) = \mathbb{D}_t \le R_t$

<u>Known results about</u> R_t <u>and</u> T_d

$R = R_1$: $t = 1$, $\mu = 0$, $\sigma = 1$

W. Feller (1951) got for $f_R(x) = \dfrac{d\mathsf{P}(R \le x)}{dx}$, $x > 0$, the following formula:

$$f_R(x) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-\frac{k^2 x^2}{2}}.$$

Remark. If $b(t) = B_t - tB_1$, $t \le 1$, is a Brownian bridge, then for Kolmogorov's distribution $F_{\mathsf{K}}(x) = \mathsf{P}\Big(\sup_{t\le 1} |b(t)| \le x\Big)$ we have

$$F_{\mathsf{K}}(x) = 1 - 2\sum_{k=1}^{\infty} (-1)^{k-1} e^{-2k^2x^2}$$

$$= \frac{\sqrt{2\pi}}{x} \sum_{k=1}^{\infty} e^{-(2k-1)^2\pi^2/x^2}$$

$$\Downarrow \quad (\theta\text{-function})$$

$$f_{\mathsf{K}}(x) = 8x \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-2k^2x^2}$$

Together with

$$f_R(x) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 e^{-\frac{k^2x^2}{2}}$$

we get

$$f_R(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{x}\, f_{\mathsf{K}}(x),$$

so

$$\mathsf{E}R = \sqrt{\frac{8}{\pi}} \quad (= 1.5957691216\ldots)$$

Theorem 1. $(t = 1,\ \underline{\underline{\mu = 0}},\ \sigma = 1.)$

(a) $$\boxed{\mathbb{D}_1 \overset{\text{law}}{=} \max_{0 \le t \le 1} |B_t|}$$

(b) *If* $F_{\mathbb{D}_1}(x) = \mathsf{P}(\mathbb{D}_1 \le x)$ *then* (*it is well known*)

$$F_{\mathbb{D}_1}(x) = 1 - \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \int_{-x}^{x} \Big[e^{-\frac{(y+4kx)^2}{2}} - e^{-\frac{(y+2x+4kx)^2}{2}} \Big] dy$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-\frac{\pi^2(2n+1)^2}{8x^2}}$$

(c) $\mathsf{E}\mathbb{D}_1 = \mathsf{E} \max_{0 \le t \le 1} |B_t| = \sqrt{\frac{\pi}{2}}$

$(\mathsf{E}\mathbb{D}_T = \sigma\sqrt{T}\sqrt{\frac{\pi}{2}})$

Proof. (a): Denote

$$M_t = \max_{s \le t} B_s, \qquad L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I(|B_s| \le \varepsilon)\, ds.$$

By Lévy's theorem

$$(M_t - B_t, M_t; t \le 1) \overset{\text{law}}{=} (|B_t|, L_t; t \le 1).$$

Hence

$$\begin{aligned} \mathbb{D}_1 &= \max_{0 \le s \le s' \le 1} (B_s - B_{s'}) \\ &= \max_{0 \le s' \le 1} \Big(\max_{0 \le s \le s'} B_s - B_{s'} \Big) \\ &= \max_{0 \le s' \le 1} (M_{s'} - B_{s'}) \overset{\text{law}}{=} \max_{0 \le t \le 1} |B_t|. \end{aligned}$$

Proof. (c): We give two proofs. Let $\beta = (\beta_t)_{t \geq 0}$ be a Brownian motion. From self-similarity

$$(\beta_{at}; t \geq 0) \stackrel{\text{law}}{=} (a^{1/2}\beta_t; t \geq 0).$$

So if $s_1 = \inf\{t \geq 0 : |\beta_t| = 1\}$, then

$$\begin{aligned} \mathrm{P}\Big(\sup_{0 \leq t \leq 1} |\beta_t| \leq x\Big) &= \mathrm{P}\Big(\sup_{t \leq 1} |\beta_{t/x^2}| \leq 1\Big) \\ &= \mathrm{P}\Big(\sup_{t \leq 1/x^2} |\beta_t| \leq 1\Big) \\ &= \mathrm{P}\Big(s_1 \geq \frac{1}{x^2}\Big) = \mathrm{P}\Big(\frac{1}{\sqrt{s_1}} \leq x\Big), \end{aligned}$$

i.e.,

$$\sup_{t \leq 1} |\beta_t| \stackrel{\text{law}}{=} \frac{1}{\sqrt{s_1}}.$$

The normal distribution property

$$\sqrt{\frac{2}{\pi}}\int\limits_0^\infty e^{-\frac{x^2}{2\sigma^2}}\,dx=\sigma$$

$$\Downarrow$$

$$\mathsf{E}\,\mathbb{D}=\mathsf{E}\sup_{0\le t\le 1}|\beta_t|=\mathsf{E}\frac{1}{\sqrt{s_1}}=\sqrt{\frac{2}{\pi}}\int\limits_0^\infty \mathsf{E}e^{-\frac{x^2 s_1}{2}}\,dx$$

We have $\mathsf{E}e^{-\lambda s_1}=\dfrac{1}{\cosh\sqrt{2\lambda}}$. Hence

$$\mathsf{E}\,\mathbb{D}=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{dx}{\cosh x}=2\sqrt{\frac{2}{\pi}}\int\limits_0^\infty\frac{e^x\,dx}{e^{2x}+1}$$

$$=2\sqrt{\frac{2}{\pi}}\int\limits_1^\infty\frac{dy}{1+y^2}=2\sqrt{\frac{2}{\pi}}\arctan(x)\Big|_1^\infty$$

$$=2\sqrt{\frac{2}{\pi}}\frac{\pi}{4}=\sqrt{\frac{\pi}{2}}\qquad\Rightarrow\qquad \mathsf{E}\,\mathbb{D}=\sqrt{\frac{\pi}{2}}$$

Second proof of the equality $\mathsf{E}\mathbb{D} = \sqrt{\frac{\pi}{2}}$ is based on the fact that

$$\sup_{t\le 1} |\beta_t| \overset{\text{law}}{=} \frac{1}{2}\int_0^1 \frac{du}{R_u^{(2)}},$$

where $R_s^{(2)}$ is a Bessel-2:

$$R_s^{(2)} = \widehat{\beta}_s + \frac{1}{2}\int_0^s \frac{du}{R_u^{(2)}}.$$

Thus,

$$\mathsf{E}\,\mathbb{D} = \mathsf{E}\sup|\beta_t| = \mathsf{E}R_1^{(2)} = \mathsf{E}\sqrt{\xi_1^2+\xi_2^2} = \sqrt{\frac{\pi}{2}},$$

$$\xi_1 \perp\!\!\!\perp \xi_2, \quad \xi_i \sim \mathcal{N}(0,1).$$

Theorem 2. $(t = 1,\ \mu = 0,\ \sigma = 1,\ \overline{\mathbb{D}}_1 = B_{\sigma_1} - \min_{\sigma_1 \le s' \le 1} B_{s'}.)$

(a) $\overline{\mathbb{D}}_1 \overset{\text{law}}{=} \sup\limits_{g_1 \le s \le 1} |B_s|$, *where*

$$g_1 = \sup\{t \le 1 : B_t = 0\}.$$

(b) $f_{\overline{\mathbb{D}}_1}(x) = \sqrt{\frac{8}{\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k e^{-\frac{k^2 x^2}{2}}, \quad x > 0.$

(c) $\mathsf{E}\overline{\mathbb{D}}_1 = \sqrt{\frac{8}{\pi}} \log 2\ (= 1.1061\ldots),$

$$\begin{array}{ccccc} \mathsf{E}\overline{\mathbb{D}}_1 & \le & \mathsf{E}\mathbb{D}_1 & \le & \mathsf{E}R \\ \sqrt{\frac{8}{\pi}} \log 2 & \le & \sqrt{\frac{\pi}{2}} & \le & \sqrt{\frac{8}{\pi}} \\ 1.1061\ldots & \le & 1.2533\ldots & \le & 1.5957\ldots \end{array}$$

Proof (a) By Lévy's theorem

$$\begin{pmatrix} M_t - B_t, M_t, B_t; \\ t \le 1 \end{pmatrix} \overset{\text{law}}{=} \begin{pmatrix} |B_t|, L_t, L_t - |B_t|; \\ t \le 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} M_t - B_t, M_t, B_t; \\ \sigma_1 \le t \le 1 \end{pmatrix} \overset{\text{law}}{=} \begin{pmatrix} |B_t|, L_t, L_t - |B_t|; \\ g_1 \le t \le 1 \end{pmatrix}$$

where $\sigma_1 = \min \left\{ s \le 1 \, : \, B_s = \max_{u \le 1} B_u \right\}$.

Therefore

$$\Big(B_{\sigma_1}, \max_{\sigma_1 \le t \le 1} (M_t - B_t - M_t)\Big)$$

$$\overset{\text{law}}{=} \Big(L_{g_1} - |B_{g_1}|, \max_{g_1 \le t \le 1} (|B_t| - L_t)\Big)$$

$$= \Big(L_{g_1}, \max_{g_1 \le t \le 1} |B_t| - L_{g_1}\Big)$$

(since $B_{g_1} = 0$ and $L_t = L_{g_1}$ for $g_1 \le t \le 1$).

Finally,

$$\overline{\mathbb{D}}_1 = B_{\sigma_1} - \min_{\sigma_1 \le t \le 1} B_t = B_{\sigma_1} + \max_{\sigma_1 \le t \le 1} (-B_t)$$

$$= B_{\sigma_1} + \max_{\sigma_1 \le t \le 1} (M_t - B_t - M_t)$$

$$\overset{\text{law}}{=} L_{g_1} + \max_{g_1 \le t \le 1} |B_t| - L_{g_1} = \max_{g_1 \le t \le 1} |B_t|$$