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Follow the Leader

Steady State Analysis of a Dynamic Social Network

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Abstract

A social pyramid is show to be the unique steady state social structure when agents gain utility from being early adopters of subsequently popular trends. The environment is related to a majority game, but introduces the importance of the timing of adoption. Utility derived from making a popular choice independent of timing is demonstrated as essential to support the hierarchy. The proposed environment is relevant to a number of settings in which leadership and timing of decisions are important or where being perceived as a trend setter is rewarded. The leadership position can be self-reinforcing. For a professional critic, for example, a cult-of-personality can dictate popular tastes, such as in art, food, and wine markets. A social hierarchy can also apply to the introduction of new products or ideas including academic research and financial market analysts.

Keywords: Dynamic Network, Social Interaction, Consumer Choice

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1. Introduction

The objective of this exercise is to model decisions by a population for a product in which tastes are uncertain and possibly endogenous to the process. Opinion leaders (Gurus) arise naturally from the population and contribute to the agents' decisions, possibly by shaping tastes.

Consider three scenarios under which consumers seek input from outside experts:

1. The consumer has a fixed set of preferences but imperfect information concerning the available product options. The expert offers information or advice that helps the consumer buy the product that maximizes his exogenous utility. The environment could be modeled as one in which the consumer possesses a highly noisy signal of the utility he or she would derive from each product and the expert, through nature or the expenditure of resources, can reduce the noise associated with each product.

2. The consumer possesses some innate preferences over the available products, but can be influenced by the opinion of others (peers) and experts. The "expert" may be someone possessing both better information than the general public about the consumer options (as in case I), but also someone who can provide advice that is consistent with the underlying preferences of a group of consumers. This situation offers a clear opportunity for feedback between the advice provided by the expert and the choices made by the population of consumers if the expert wishes to be an opinion leader with a wide range of influence.

3. The consumer has no innate preferences. The consumer's tastes are fully fashioned by the influence of peers and experts. In this case, the expert shapes opinion, but need not have any special advantage in evaluating the options. Success is derived from the expert's influence over general population's choice.

To motivate the endogeniety of preferences, consider the purchase of wine. The wine critic plays an integral role in the purchasing decisions of many consumers. The wine critic can be a professional who makes a living rating and reporting about wines, it may be the helpful proprietor or employee at a preferred wine retailer, or it may be an acquaintance of the consumer who is perceived to be better informed about wines.

Preferences for certain products, including wine, are clearly subjective. For many such products, the consumer may be able to recognize difference, but may seek guidance in ranking these differentiated products. It seems quite reasonable that a consumer, informed by an expert that a particular bottle of wine is of high quality, will then adapt his definition of "good" to accommodate the flavors experienced. The fact that a particular description of the wine's character is provided, for example "Red purple hue. Rich, ripe dark fruit aromas follow through with concentrated and mouthcoating flavors of black cherry and tobacco. Finishes with very rich tannins and well-integrated oak."[1] fails to explain why this particular set of flavors deserves the particularly high ranking or why "Brilliant garnet-ruby red hue. Cherry, strawberry and cedar aromas. A medium-bodied palate leads to a simple, clean finish with bright fruit, soft tannins and balanced acidity."[2] scores lower. Consumers who follow the advice may very well learn (rather than innately believe) that black cherry and tobacco with tannins and oak are superior flavors to cherry, strawberry and cedar.

Consider the ranking of the chateaus of the Bordeaux region of France. In 1855, Napoléon III assigned wine merchants the task of ranking wines for the upcoming Exposition Universelle de Paris. Without holding tastings, the merchants selected roughly 200 top chateaus of the region, bestowing upon them the label "Classified Growth." Further indication of quality was designated by dividing the group into five categories, "first growth" (the best) through "fifth growth." Original chateaux selection and ranking was based on reputation and the price of their wines.[3] The list has remained largely fixed to the present.[4] More importantly, so has the perception of quality despite numerous changes in ownership, management, production methods, technology, and the expansion of grape growing and wine production to many new regions of the world. Until recently, wine critics have been accused of adjusting their pallet each year in order to grant the most favorable reviews to the wines produced by the first growth chateaus. Such behavior by critics and the acceptance of such behavior by consumers suggests that preferences

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[1] Wineanswers.com online review of Lagrange 2000 St. Julien (score: 93 points, exceptional) http://www.tastings.com/scout_wine.lasso?id=165665

[2] Wineanswers.com online review of Cave de Chusclan 1999 Chateau de Gicon, Côtes-du-Rhône Rouge (score: 84 points, recommended) http://www.tastings.com/scout_wine.lasso?id=165925

[3] This original ranking could be deemed a reflection of some exogenous measure of quality. The price was believed to be an accurate reflection of the wine's quality. Marketing, technology, and economies of scale were fairly homogenous. Alternatively, consistent with this paper, this original list could be considered arbitrary.

[4] The only change was in 1973 when Château Mouton-Rothschild was elevated from a second growth to a first growth vineyard.

are fungible. Today's wine critics are perceived as applying a more consistent standard to their wine tasting, but the ability of consumers to learn to enjoy what they are advised to appreciate through acquisition of an educated palate remains.

The rise of Robert Parker Jr. as a leading wine critic, whose influence is recognized as dominant in the industry, raised alarm among other wine experts and among the wine production industry. The concern is that Parker's dominant position imposes uniformity in consumption and tastes.[5] Parker's dominance is evidenced by the belief, true or not, that certain wine producers have been thought to produce wines specifically intended to conform to Parker's tastes in order to achieve a high rating. Under case I or II, homogeneous tastes among the wine consuming public would explain Parker's dominance. In this scenario, Parker's dominance over other wine critics reflects Parker's ability to offer the most accurate signal on wine quality (cases I and II) consistent with tastes (cast II only). Alternatively, under case III consumers tastes are malleable to conform to Parker's advice. The reward to drinking wine is from drinking wine that is recognized by the consumer and his or her peers as high quality wine (based on Parker's recommendation).

This project focuses on issues concerning the social phenomenon by which an agent in the population becomes a leader who shapes the choice of others in the population. Within this examination is the question of whether the leadership position and structure of the social network of the followers is stable. Examinations of the feedback relationship between a leader and his or her followers and other social phenomenon related to this environment will be left for other examinations.

Social networks have recently received considerable attention in economics for explaining aggregate social behavior. In these models, individuals may be influenced globally by the aggregate behavior of the population or locally by their neighbors, a subset of the population who serve as a reference group. This project incorporates aspects of both. Economic incorporation of local interactions in decision making include Schelling (1971) and (1973) and Katz and Shapiro (1985). The network is a social structure by which discrete decisions are made or new products or technologies are adopted. The utility of a choice depends on the choices of

[5] A number of profiles of Parker exist, including *The Atlantic Monthly* (2000).

those in an agent's peer group. Because of social interactions, products that garner no particular consumer preference, and may even be perceived to be inferior, can grow to dominate consumer choice. Social interactions that produce a majority payoff have been used to explain the persistent use of the QWERT typewriter key arrangement, the dominance of VHS over Beta video tapes and DOS over Macintosh operating systems. Analysis of these models has been facilitated by employment of tools from statistical mechanics to explore stability, multiple equilibria, and switching.

The reward structure for this project moves away from the simple majority or minority payoff to take timing into account. Agents are rewarded for being early adopters of a trend that subsequently becomes popular. Financial or social reward come to those who can gain a reputation for being a predictor (or setter) of trends. "Wine geeks just love bragging rights. They get kudos from their peers when they get a high-score wine first or get it cheaper" (Bialek quoted in *Los Angeles Magazine*, 1998(Dec)). Because the agents seek reward through early adoption, timing of an agent's decision must be modeled as an endogenous component of the social network, a clear deviation from the traditional mean field examinations of local interaction. Further, agents are rewarded based on aggregate behavior rather on the local behavior of the individual's peer group. Brock and Durlauf (2001) also models the individual as seeking to conform to the social norm, but in an environment in which the payoff is based purely on the aggregate behavior. They model decisions made simultaneously and rewards are according to expectations of the norm rather than its realization.

The traditional examination of social interactions in economic decisions is based on a fixed social network. Regular structures are convenient for deriving aggregate properties of the network. The level of connectedness, for example, determines the rate at which information disperses through the network, which can impact the leader and follower structure.

Dynamic models such as Watts (2001), Bala and Goyal (2000), Jackson and Watts (2002), and Kirman *et al* (2007) are concerned with evolution and convergence. Bala and Goyal (2000) examine the evolution of a network in which agents create links with individuals with whom the benefit exceeds the cost of maintaining the link. Each agent *i* offers a direct link benefit (that is uniform), but also the benefit gained by providing indirect access to those with whom agent *i* is linked. Thus, the benefit of linking to an individual is endogenous to how connected that individual is to others in the population. One stable attracting network configuration is a star formation where one agent serves as a hub through which all agents connect. The behavior of the central agent is thus highly influential, but the agent is not a leader, but a follower attuned to the actions of everyone else in the network.

Networks are also a vehicle for information transmission. Directed networks (where the link between two agents is established by one of the linked pair) have been useful for examining advertising and marketing, as in Dutta and Jackson (2000). Undirected networks (where the link requires the support of both agents in a linked pair) also serve as a mechanism of information dispersion, as in Ellison and Fudenberg (1995) where word-of-mouth communication can lead to conformity in behavior.

Information, evolution, convergence, and stability are all features of the proposed project, but the payoff structure and its dependence on the timing of a decision substantially and substanatively alter the role played by the social connections, thus altering the process of formation and self organization. The directional path through which information is transmitted gives the network a hierarchical structure with a leader and follower. That the directed links are established by agents seeking information rather than by advertisers looking to push information is another important distinction affecting the formation of the network, its employment by the agents, and the behavior that it tends to capture.

Kirman *et al* (2007) point out two relevant deficiencies in the earlier dynamic models. In setting the mechanism by which links are updated, one deficiency is that each link, existing or potential, has a value known to the agent. This knowledge required that the agent know the existing structure of the full social network and the mapping from the network structure to payoffs. A second concern is with the artificial process determining the order with which agents update their links.

Like this examination, Chang and Harrington (2005) consider an evolutionary network. Their population is heterogeneous with some innately better suited to innovation and other innately better suited to dissemination of ideas. They examine the endogenous social network that arises to connect these different types of researchers that results in a symbiotic relationship between innovators and those who are able to facilitate in the propagation of the innovator's ideas. For the current investigation, the leader/follower type behavior results from the agent adapting to his or her endogenously determined environment in a manner to improve their payoff.

The model is developed analytically to demonstrate the existence and stability properties of a leader/follower social hierarchy. Simulations explore those features that facilitate or hinder the emergence of the social hierarchy from a uniform or random initial social network.

2. Model

To more easily relate to the sequence of events in the model, consider high-school students deciding on what fashion statement to make a school. At the beginning of each week, there are a number of fashion options to choose from.[6] Each student wants to make a fashion choice that is popular. In addition, each student wants to be an early adopter of trends that become popular in order to be perceived as either attuned to or influential in social trends. Some students choose to make an independent choice, arriving to school on Monday displaying their decision. Others wait to see what their friends and acquaintances are doing. This gives them a better chance of ending up in the majority, though at the opportunity cost of not being perceived as a leader.

2.1 Properties of a hierarchy

The properties of the hierarchy can be examined in a static environment. Consider a population of *L* agents. Each agent $i \in \{1, 2, 3, ..., L\}$ chooses from among *K* available options, $k \in \{1, 2, 3, ..., K\}$. Agent i's choice is captured in notation by x_i. The choice represents the action of adopting a product, fad, or fashion statement for the given period. The *K* options are new to the agents so that they have no prior experience with these particular options.

To inform his decision, agent *i* maintains *d* one-way (directed) non-redundant links to other agents of the population. If agent *i* maintains a link to agent *j*, then *j* is one of agent *i*'s "friends." In the context of the decision process, agent *i*'s link to agent *j* allows agent *i* to observe the

[6] The source of the options is not essential, but in the current narrative, they could have been presented as possibilities by the fashion seen on popular television programs during the preceding week or in the advertisements in the latest issue of a weekly fashion magazine.

choice made by agent *j* at the time the choice is made. The link allows agent *i* to imitate the behavior of agent *j* if he so chooses. The target of any link is unaware of the presence of the link and thus the relationship between agent i and agent j is reciprocated only if agent *j* independently maintains a link to agent *i*.

The decision regarding the adoption of *k* proceeds in rounds, $r \in \{1, 2, 3, ..., R\}$. Prior to the first round, each agent must make a decision about whether to act independently when selecting from the *K* options or whether to wait to imitate the choice made by one of their friends. Those who act independently will choose their x_i in the first round. Those who choose to imitate make there choice in the round following the round in which they observe their friend's choice. Those who choose to imitate a friend are designated as "followers." A "leader" is an agent who acts independently and whom other agents have chosen follow. An agent who acts independently but without followers is not a leader.

This information is sufficient to generate a structured leader/follower hierarchy.

Proposition 1: Consider a network of *L* agents, each occupying a single node. Assume that each agent is link to each of the other $L-1$ agents either directly or indirectly through a chain of linked nodes. For every agent *i*, there exists a structured hierarchy based on agent *i* as the leader and every agent *j*, $j \neq i$ as a follower.

Proof: Let Tier 0 of the hierarchy, the leadership position, be occupied by agent *i*. Let Tier δ of the hierarchy, $\delta = \{1, 2, 3, \ldots, \Delta\}$ be populated by agents possessing a direct link to a member of tier $\delta - 1$ and without links to members of Tier 0 for $\delta = 1$ and Tiers 0 through $\delta - 2$ for $\delta > 1$. Tier Δ is the final tier required to link each of the $j \neq i$ agents to agent *i*. The hierarchy exists by construction.[7]

Let a hierarchy be considered efficient if each agent within the hierarchy occupies the highest tier (lowest numbered tier) available to that agent. A hierarchy will be inefficient if there

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[7] It would probably be simple enough to drop the assumption of a fully linked network and demonstrate that the probability that no route of linkages connect agent *i* to agent *j* approaches zero as the number of links increase. (I was able to prove that the probably that an agent *i* has no incoming links goes to zero as *d* increases and is effectively zero for *d*>=4 independent of *L*.)

exists agent $j \neq i$ who could occupy Tier δ_1 but instead occupies Tier δ_2, $\delta_2 > \delta_1$ as a result of imitating a friend who is not the best positioned of their friends in the network.

Proposition 2: For each agent *i*, there is a unique efficient hierarchy based on agent *i* as the leader.

Proof: True by construction. Efficiency flows down from the top of the hierarchy. An efficient hierarchy has Tier 1 is occupied by all agents with one of their *d* links pointed at agent i. This set is unique, depending only on agent *i* occupying Tier 0. The argument applies to each subsequent tier in the hierarchy so that each set of agents efficiently occupying Tier δ is unique as determined by the agents occupying Tier $\delta - 1$.

Propositions 1 and 2 establish the existence of a unique efficient hierarchical social structure based on agent *i* as the leader. The next step, determining whether the followers wish to follow, requires specification of a payoff function.

Let the utility payoff to agent *i* derived from the adoption of *k* depend on three components,

$$\pi_i = u(x_i) + T(N_{ki}^R - N_{ki}^r) + J(N_{ki}^R) \quad (1)$$

The first term, $u(x_i)$ is the individual's innate private utility derived from the choice. This particular investigation is limited to the special case of $u(x_i) = 0$, which will be presumed hereafter.

$$\pi_i = T(N_{ki}^R - N_{ki}^r) + J(N_{ki}^R) \quad (2)$$

Assigning $u(x_i) = 0$ implies that there are no innate differences between the available options. Each agent who chooses to select from the available options independently selects one of the available *K* options with probability $1/K$ each.

The $T(N_{ki}^R - N_{ki}^r)$ element of the payoff rewards the agent for being an early adopter of a subsequently popular trend. Here, N_{ki}^R represents the total number of agents to have chosen option *k* by the final round of decision making, *R*. The inclusion of *i* in the subscript indicates that the agent is among those agents in the population to have chosen this particular *k*. The *i* may be dropped when inclusion is not necessary. The second determinate, N_{ki}^{r}, represents the total number of agents to have chosen option *k* by round *r*. Thus, $N_{ki}^{R} - N_{ki}^{r} \geq 0$ represents to total number of agents who have adopted choice *k* that coincides with that of agent *i* in rounds subsequent to agent *i*'s adoption. Accordingly, the function *T* assigns a reward to agent *i* based on the number who adopt the same choice *k* in subsequent rounds with $T(x) \geq 0$, $T(0) = 0$, and $T'(x) \geq 0$.

The $J(N_{ki}^{R})$ component captures the social interaction term, rewarding the agent for the popularity of a choice regardless of the round in which it was adopted, with $J(x) \geq 0$, $J(1) = 0$ and $J'(x) \geq 0$

Consider a hierarchy with agent *i* as the leader. Further, consider the payoffs received if every agent participates in the hierarchy by imitating agent *i*, directly or indirectly. With full participation, $N_{ki}^{R} = L$. Every agent in the population receives the a social component of $J(L)$. For those agents occupying the lowest tier of the hierarchy $\pi_{j} = J(L)$ since for these agents $N_{kj}^{r} = N_{kj}^{R} = N$ so that $T(N_{kj}^{R} - N_{kj}^{r}) = 0$. Those agents occupying tiers above the lowest tier receive a total payoff that increases with their distance from the bottom. At the top, agent *i* receive the payoff $\pi_{i} = T(L-1) + J(L)$.

Consider the options available to an agent *j* who occupies the lowest tier of the hierarchy. Agent *j* must decide whether it is better to participate in the hierarchy, occupying his position at the bottom, or to independently make his own decision about which *k* to choose. As noted, agent *j*'s position at the bottom of the hierarchy earns him a payoff of $\pi_{j} = J(L)$. Assuming continued participation in the social hierarchy by the remainder of the population, acting independently offers agent *j* one of two possible payoffs. The good outcome for agent *j* is if he coincidently chooses the same as agent *i*. In this case, agent *j* is awarded a payoff

$$\pi_{j} = T(L-2) + J(L).$$

The bad outcome is to have chosen one of the options not coincidently selected by the leader. In this case, the payoff is

$$\pi_j = J(1) = 0 .$$

2.2 Hierarchy as Equilibrium

Condition A: $J(L) \geq \frac{1}{K}\left(T(L-2)+J(L)\right)+\frac{K-1}{K}J(1)$

Since $J(1)=0$, Condition A can be expressed more succinctly, but keeping the last term is illustrative.

Proposition 3: Assume $L \geq 3$ risk neutral *ex-ante* homogeneous agents, differentiated only by their position in the social network. If Condition A is met for the hierarchy with agent *i* as the leader, then the full participation by the population is an equilibrium.

Proof: For $L \geq 3$ the right-hand-side of the inequality in Condition A is the expected payoff to the non-leader for acting independently. If the non-leader chooses to imitate, then $N_{ki}^{R} = L$. It is the same for every follower in the social hierarchy. The left-hand-side of the inequality is the payoff earned by a follower occupying the lowest tier of the social hierarchy. If the non-leader chooses to act independently, and selects the same as the leader, $N_{ki}^{R} = L$ and $N_{ki}^{1} = 2$. If independent action leads to a choice other than that made by the leader, then $N_{kj}^{R} = N_{kj}^{r} = 1$. If Condition A is true, those agents occupying the lowest tier of the social hierarchy will choose to participate. By extension, a favorable payoff that induces the occupants of the lowest tier to participate is sufficient to also induce participation by those at higher tiers.

Proposition 3 is based on the comparison of payoff for participating in the social network versus acting independently, given the continued participation by the remainder of the population. If the agent in the works position in the social network in better off imitating rather than acting independently, then the same is true for those in better positions, receiving higher payoffs for their participation. Relaxing the assumption of continued participation by other non-leaders in the hierarchy will be considered later.

Rearrange Condition A to reveal a minimum number of choice options to maintain full participation of the population.

$$K \geq \underline{K} = \frac{T(L-2)}{J(L) - J(1)} + 1 \tag{3}$$

For *K* below the bound, the favorable probability of choosing the same as the leader is enough to induce them to gamble with acting independently rather than accepting the known social payoff.

In (2), the numerator, $T(L-2)$, is the potential gain to agent *j* if he gambles and wins by coincidently selecting the same choice as the leader. The denominator, $J(L) - J(1)$, is the differential between the certain payoff from following the group and the bad state of acting independently. It thus represents the potential loss of failing to coordinate with the leader.

A realization of $\underline{K} < 2$ ensures that full participation in the leader/follower hierarchy is an equilibrium as long as there is a choice. Having more than one option creates the possibility of choosing "wrong" when trying to coordinate independently with the leader. If $K \geq \underline{K}$ then because the agent risks losing the social component without sufficient reward for the risk.

From (2), low $T(L-2)$ or high $J(L)$ encourage followers to remain in the hierarchy, as reflected by the resulting low $\underline{K}$. If the reward to being an early adopter is low, then there is little incentive to risk the sure social payment of $J(L)$. A large $J(L)$ raises the downside cost of the gamble that goes with acting independently.

Rearranging (2) to solve for a_1 gives

$$J(L) \geq \frac{T(L-2)}{K-1}.$$

Thus, to maintain the hierarchy as an equilibrium requires a minimum social payoff. The more options available to the agents, the lower the social payoff required to maintain the equilibrium due to the increased likelihood of ending up as a solo adopter for their choice.

2.3 Stability

If agent *j* occupying the lowest tier of the hierarchy chooses to act independently, this changes the payoff received by all of the remaining participants. The timing component drops from $T(N_{ki}^R - N_{ki}^r)$ to $T(N_{ki}^R - 1 - N_{ki}^r)$ and the pure social payoff drops from $J(N_{ki}^R)$ to $J(N_{ki}^R - 1)$. The choice to act independently by agent *j* also changes the expected payoff of acting independently for the other agents in the hierarchy since there is a chance of coincidently choosing the same option. A concern, then, is whether a decision by agent *j* to act independently, whether rational or irrational, causes a cascade of other participants to choose to also act independently. At issue is the stability of the hierarchy. To evaluate, introduce the notion of fractional participation.

Let λ be the proportion of the population that participates in the social hierarchy. The population of $L(1-\lambda)$ potential followers acting independently, randomly selecting among the *K* options, means that N_k^R and N_k^r are random variables for $k = 1,\ldots,K$. The relevant condition that determines whether the agent at the bottom of an existing hierarchy is expressed as follows:

Condition B:

$$E(J(N_{ki}^R) \mid x_j = x_i) \geq \frac{1}{K} E\left(T(N_{ki}^R - N_{ki}^1) + J(N_{ki}^R) \mid x_j = x_i\right) + \frac{K-1}{K} E(J(N_{k(-i)}^R) \mid x_j \neq x_i)$$

Consider the following linear functional forms of *T* and *J*.

$$T(N_k^R - N_k^r) = a_0(N_k^R - N_k^r) \qquad (4)$$

$$J(N_k^R) = a_1(N_k^R - 1). \qquad (5)$$

In this case, $\underline{K}$ in (2) can be expressed using the following:

$$\underline{K} = \frac{a_0(L-2)}{a_1(L-1)} + 1 = \frac{a_0}{a_1} + 1 - \varepsilon \qquad (6)$$

where $\varepsilon > 0$ and is small for large *L*.

The coefficients a_0 and a_1 differentiate the payoff to agent *j* for being an early adopter from the payoff for making a popular choice. For $\underline{K}$ to be a meaningful boundary requires $a_0 > a_1$, which may be deemed natural for the products being considered. For any $a_0 > 0$ normalizing $a_0 = 1$ leaves a_1 as the down weighting to given to the social payoff relative to the early adoption payoff.

Consider agent *j* who occupies the lowest populated tier of the hierarchy. The following are the expectations are used to evaluate agent *j*'s options (λL includes agent *j*):

$$E(N_{ki}^{R} \mid x_j = x_i) = \lambda L + (1-\lambda)L/K$$
$$E(N_{ki}^{r} \mid x_j = x_i, r_j = 1) = 2 + (1-\lambda)L/K \quad (7)$$
$$E(N_{ki}^{R} \mid x_j \neq x_i) = 1 + (1-\lambda)L/K.$$

Here, Use the expectations in (6) in (3) and (4), Condition B can be expressed as:

$$(\lambda L - 1)(a_0 - a_1(K-1)) \le a_0 . \quad (8)$$

Let λ^* be the value of λ for which (7) holds with equality,

$$\lambda^* = a_0 / L(a_1(K-1) - a_0).$$

Proposition 4: Given a linear payoff as described in (3) and (4),

a) If $a_0 - a_1(K-1) < 0$ then $\lambda = 1$.
b) If $a_0 - a_1(K-1) > 0$, then $\lambda = \lambda^*$ with $\lambda^* \in (0,1]$.
c) If $a_0 - a_1(K-1) = 0$, then λ is unspecified.

Proof: Consider the hierarchy in which λL agents are participatory in the hierarchy (including agent *i*, the leader. The remaining $(1-\lambda)L$ potential followers in the hierarchy choose *k* independently. Agent *j* occupies the lowest tier of the λL agent hierarchy. Agent *j* participates in the hierarchy if the partial inequality in (7) holds. For $a_0 - a_1(K-1) < 0$, then (7) can be expressed as $\lambda \ge \lambda^*$. Since $\lambda^* < 0$ in this case, agent *j* participates for all values of λ including $\lambda = 1$. Therefore all potential followers will choose participation over independent action. For $a_0 - a_1(K-1) > 0$, the criteria for participation by agent *j* as expressed in (7) becomes $\lambda \le \lambda^*$. Thus, a stable equilibrium is characterized by participation in the social hierarchy by the $\lambda^* L$ occupying the upper tiers of the hierarchy. The remaining $(1-\lambda^*)L$ of the population who would otherwise occupy the lower tiers of the hierarchy choose to act independently in choosing *k*.

The condition required for full participation under Proposition 4a is consistent with (5), the condition on *K* that also produces full participation. The interior population proportion described in Proposition 4b arises when the conditions generating full participation are not satisfied.

The finding that the interior λ^* increases as a_0 / a_1 increases, increasing the expected payoff to early adoption is consistent with the finding for $\underline{K}$ in (2). For the hierarchy to exist there needs to be both a choice among the *K* options and a social payoff. If $a_1(K-1) = 0$ then $\lambda^* = 1/L$, indicating that each agent acts independently.

For $a_1 = 0$, there is no social payoff and thus zero payoff to those agents in the lowest tier of the hierarchy. For non-zero a_1, there is a possibility of a positive payoff for acting independently. Since every agent will opt to choose to act independently rather than exist in the lowest tier of the hierarchy, the hierarchy cannot exist. If only one option is available with $K = 1$, then there is no need to imitate.

The fact that λ^* is independent of *L* and of *d* means that the participation rate among the population is independent of the population size and of the number of links maintained by each agent.

2.4 Uniqueness of the hierarchy

The possibility of multiple coexisting hierarchies must be examined. The possibility that for $\lambda = 1$, the population sorts into multiple structured hierarchies, each with a different leader has to be confirmed or eliminated. Likewise, for $\lambda < 1$, the possibility that those who would otherwise act independently do not instead form into an alternate hierarchy under a different leader must also be examined.

Consider two hierarchies. Agent i_A leads hierarchy *A* and agent i_B leads hierarchy *B*. The remaining agents select whether to participate in hierarchy *A*, hierarchy *B*, or to act independently. Consider a sequential process to place the agents into one of the two hierarchies. All of the agents with a direct link to agent i_A or agent i_B occupy tier $\delta_A = 1$, the top tier of hierarchy *A*, or tier $\delta_B = 1$, the top tier of hierarchy *B*, respective. In general, the occupants of tier δ_k, $k = A, B$ are unable to link to any agents occupying tiers 1 through $\delta_k - 2$, $k = A, B$ and are able to link to an agent in tier $\delta_k - 1$. Proposition 2 does not hold in the presence of multiple simultaneous hierarchies if an agent is able to occupy the same tier in more than one hierarchy. Let the population of those able to occupy the same tier of both hierarchies divide evenly between the two hierarchies (with any odd numbered participant choosing with equal probability). Iterate the process until every agent in the population is assigned to one of the two hierarchies.

The possibility of multiple equilibrium hierarchy is evaluated based on examining the options available to those agents at the bottom of the hierarchy. As when considering a single hierarchy, those populating the lowest active tier of a hierarchy are those with the weakest incentive to remain a participant of the hierarchy.

Proposition 5: If a hierarchy exists in equilibrium, then:

a) the existence of multiple hierarchies depends on the presence of limits on the ability of agents to choose from all available options.
b) the Pr(multiple hierarchy) decreasing in *d*.

Proof: Consider the situation in which every agent participates in one of two existing hierarchies. Agent *j* occupies tier Δ_k at the bottom of hierarchy h. Let $d_j^h \in [0, d]$, be the number of links that agent *j* has pointing to other members hierarchy *h*. Since agent *j* occupies tier Δ_k, all d_j^h point to occupants of tier $\Delta_h - 1$ or Δ_k and all $1 - d_j^h$ links point to occupants of tier $\Delta_{-h} - 1$ or τ_{-h}, the two lowest tiers of the other hierarchy.

For hierarchy *A* larger than hierarchy *B*, $N_{kA}^R > N_{kB}^R$ and $J(N_{kA}^R) > J(N_{kB}^R)$. Every agent j_B occupying tier Δ_B at the bottom of hierarchy *B* with $d_j^h < d$ is better off using the link to the larger hierarchy *A*. If every agent j_B has $d_j^h < d$, then hierarchy *B* is unsustainable. A necessary, though not sufficient, condition for the persistence of hierarchy *B* is the existence of agents with $d_j^h = d$ (after the departure of those with $d_j^h < d$), which is the basis of 5a of the proposition. Part 5b follows from the fact that $\Pr(d_j^h = d) \to 0$ as *d* increases. The case for which $N_{kA}^R < N_{kB}^R$ is symmetric. For $N_{kA}^R = N_{kB}^R$, the two hierarchies represent an unstable equilibrium since the defection of one agent to the other hierarchy, say from *B* to *A*, increases that agent's payoff and makes the hierarchy *A* more attractive.

A sufficient condition for the persistence of the smaller hierarchy *B* is that all remaining participants, including those in 1 through $\Delta_B - 1$ also have $d_j^h = d$. Clearly, hierarchy *B* may persist even if this condition is not met. It may be that the early adoption payoff for those in the higher tiers of hierarchy *B* remains superior to their alternative lower tier of the larger hierarchy *A*. The leader of hierarchy *B*, agent i_B, will prefer joining hierarchy *A* at the lowest tier to remaining as the leader of hierarchy *B* if

$$\frac{N_B^R - 1}{N_A^R - 1} < \frac{a_1}{a_0 + a_1}. \tag{9}$$

Finally, the agents unable to switch out of hierarchy *B* may choose to act independently rather than remain in the smaller hierarchy. Consider $N_{kA}^R \geq N_{kB}^R$ for a sustainable hierarchy *B*. Using the payoff functions in (3) and (4), the minimum *K* required to maintain participation in the hierarchy is $\underline{K}_B$,

$$\underline{K}_B = \left\{ \frac{a_0}{a_1} \left(\frac{N_{kB}^R - 2}{N_{kB}^R - 1} \right) + 1 \right\} + \left\{ \frac{a_0}{a_1} \left(\frac{N_{kA}^R - 1}{N_{kB}^R - 1} \right) + \frac{N_{kA}^R}{N_{kB}^R - 1} \right\}. \tag{10}$$

For large *L* and $N_{kA}^R = N_{kB}^R$, $\underline{K}_B \approx 2\underline{K}$. The smaller N_{kB}^R relative to N_{kA}^R, the more choices are required to induce participation by those trapped in the lowest tier of the smaller hierarchy. For $K < \underline{K}_B$, the choice of tier τ_B agents to independently offers another route for the dissipation of hierarchy *B*. (end of proof, May need to end earlier, take more discussion into the body of the paper, or look for ways to shorten.)

Given a payoff structure that rewards individuals for being early adopter of a subsequently popular trend or product, a structured social hierarchy with a leader and a population of followers can be the unique equilibrium. The existence of a hierarchy depends on a sufficient reward to even those who are the last to adopt. The willing followers give support to the complete hierarchy. The uniqueness of the hierarchy requires the fully connected social network assumed in Proposition 1.[8]

[8] Though not examined here, intrinsic preferences would also introduce the possibility of multiple hierarchies.

3 Emergence

Consider the behavior of individuals as they seek to be early adopters of subsequently popular trends or products. Simulation is used to explore the emergence of a leader from a population and the nature of the social order that forms from under the leader.

3.1 Efficiency and Optimality

A hierarchy will be considered efficient if, given the leader, each agent who is a member of the hierarchy occupies the highest tier that is available to the agent through his or her links. Thus, a hierarchy that matches the equilibrium hierarchy developed in Section 2 is perfectly efficient. Deviations from the equilibrium hierarchy are inefficient. Formally,

Definition 1: Let a hierarchy be considered efficient if each agent connected to the hierarchy occupies the highest tier (lowest numbered tier) available to that agent.

If each agent has *d* links to other agents in the population, then the average number of incoming links directed at each agent is *d* as well. In the case where the *d* outbound links are assigned randomly, the number of incoming links to each agent will be a random number. Let those agents with the maximum number of incoming links be considered a natural leaders.

Definition 2: A hierarchy will be considered optimal for the social structure if the hierarchy's leader is a natural leader.

A leader may emerge who is not a natural leader, in which case the hierarchy is not considered optimal (though it may still be efficient given the leader).

3.2 Evolution in a Dynamic System

A dynamic model of socially linked individuals based on described incentives is developed in Goldbaum (2008). Briefly, in the dynamic model, agents decide whether to act independently or to imitate, and which friend to imitate, probabilistically. Dynamics are introduced to the social network through algorithms that update probabilities based on experience and observation. Updating algorithms that allow convergence towards placing full probability mass on the highest rewarding strategy confirm the analytical results, producing a single fully efficient hierarchy with full participation for $K > \underline{K}$ and appropriate levels of partial participation for $K < \underline{K}$.

Persistent randomness in the selection of which friend to imitate prevents convergence to the unique fully efficient hierarchy. A robust outcome is for a single hierarchy to emerge where the leadership position is stable, once determined, but with a fluid structure of followers that is less than fully efficient. If there is enough noise, the randomness can occasionally depose the leader, allowing a new leader to emerge. The process tends to produce increasingly fit leaders, increasing the optimality of the hierarchy. Multiple smaller stable hierarchies can emerge if the chain of links connecting the lowest tiered agents to the leader becomes too long. In this case, it is better to have a reliable link to a smaller hierarchy than an unreliable link to a larger hierarchy.

The memory length or biases towards the most recent performance observations also affect the long run behavior of the social network. The emergence of a leader from an unstructured social network requires that transitory random events become embedded in the social network through individual agent adjustments. A lucky selection by agent *i* early in the simulation has to cause others to modify behavior towards imitating the agent. Success breeds further success as agent *i* becomes empowered by his or her followers. A failure of those linked to agent *i* to respond to his or her initial success fail to generate the emergence of a leader, without which there is no hierarchy.

5. Conclusions

A singular hierarchical social structure is shown to be a natural equilibrium structure when individuals are motivated to be early adopters of a trend or behavior. Given a leader, the hierarchical structure forms naturally from each individual's incentive to follow the leader in selecting from the available options and to learn of the leader's choice through the shortest possible chain of links.

Once leadership is established, the leader is empowered by the followers. The hierarchy would collapse were it not for the willingness of some agent(s) to be at the bottom of the hierarchy, thus it is demonstrated that a "majority" component that is independent of the time of adoption is an essential element of the social reward. Without the majority component, there is no incentive for the lowest tiered individuals to imitate and the hierarchal structure collapses as a social equilibrium. A weak majority component induces those agents otherwise occupying the lowest tiers of the hierarchy act independently rather than to imitate. A small number of options to choose from produces the same behavior by providing an incentive to act independently in the hopes of making a choice consistent with the leader. In neither case does the independent behavior of these "non-leaders" threaten the stability of the leader.

5.1 Other and future applications

Other markets and products offer similar relationship between leaders and followers. Preferences (and appreciation) of art is shaped by the opinion expressed by influential art critics and educators. New art is constantly being presented to and scrutinized by the public. The marketing of new art by fledgling artists is clearly influenced by people and institutions that can shape opinion. Certain galleries and other outlets build and maintain a reputation for selecting the best new artists. Clearly, this creates an environment in which a gallery's selection of an artist as a new "hot" producer can be self-fulfilling.

Early publication in a new research area that subsequently takes off offers clear rewards to academic researchers. Success and influence are measured by the number of citations. Speed is clearly important and the involvement of a top researcher increases the likelihood that the field will be developed to become an essential and recognized area of thought. For this reason, many new academics scan publications of working paper series for potential projects. The objective is to become affiliated with a new line of research initiated by a well established researcher. The young researcher who seeks to make a contribution through her own original research may produce a work that offers greater contribution to academic knowledge, but risks that the contribution will not be recognized or developed by other academics.

Timing is important in minority game models of financial markets. The minority game, originally derived from the Arthur (1994) El Farol problem is employed by Challet *et al* (2001) to capture stylized facts in financial markets. In the model, the individual investor who buys an asset when it is a minority choice among the population of investors (and thus low priced) and sells when the asset is a majority choice (and thus high priced) will earn a positive return. Clearly, an investor who can buy into a subsequently popular investment option stands to gain as well. Subscriptions to stock picking services, internet chat rooms, and other forums may provide the needed social network for such behavioral investing. Applying the current model to financial markets seems a reasonable expression of the social network that exist in financial chat rooms occupied by day traders, characterized in Mizrach and Weerts (2006) and other forms of expert investment advice. It also offers a richer characterization of market timing that can lead to behavioral financial market returns.

Politicians must decide at what point and how actively to endorse new policy proposals. Taking an early stand offers the rewards of being affiliated with a popular initiative an the risk of association with unpopular proposal. Late endorsement is safer, but less rewarding. A senator who authors and introduces new legislation will have his or her name associated with the bill. The Bipartisan Campaign Reform Act of 2002 is also known formally and informally as the McCain–Feingold Act.

5.2 Future model extensions

5.2.1 Innate preference

Reintroduce innate preferences with $u(x) > 0$. There are many issues to explore allowing for consumer innate preferences. Once the consumers have preference over the set of options they face, then there is a need on the part of the consumer to balance the two social aspects to consumption with their own innate preference. Once consumers are defined in a preference space, there is considerable opportunity to examine how these populations organize themselves and how leaders emerge to serve a population defined by their preferences. Stability of a leader's position, the sustainable size of a hierarchy, and how broad a population a leader can influence are all rich areas for investigation.

Noise and drift in innate preferences can also be introduced. A leader's taste may be an imperfect reflection of the preferences of the population of followers. The leader may find the preferences of what was once a homogeneous group of followers has grown to be more diverse. Another modification that could act as a catalyst for change is to introduce cost for information gathering.

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