UNIVERSITY OF TECHNOLOGY SYDNEY

SCHOOL OF FINANCE AND ECONOMICS

UTS:BUSINESS

WORKING PAPER NO. 14 **JANUARY, 1992**

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ISSN: 1036-7373
http://www.business.uts.edu.au/finance/

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DEVELOPMENTS IN NONLINEAR ECONOMIC DYNAMICS: PAST, PRESENT & FUTURE

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To appear in: "Die Zukunft der Okonomischen Wissenschaft" ed. H. Hanusch, Verlag Wirtschaft und Finanzen, 1992.

1. Introduction

In this short essay I would like to review the relationship between progress in techniques for the analysis of nonlinear dynamical systems and progress in the construction and analysis of nonlinear models of economic dynamics. Given that the two strands of literature to which I refer are both now quite voluminous it is to some extent inevitable that I can only focus on a subset of nonlinear models of economic dynamics which are currently of interest. My choice of models is to some extent governed by my own research interests but I believe covers a sufficiently broad range to also be of interest to many economists working in economic dynamics.

The study of models of economic dynamics, in the form of the literature on business cycle fluctuations has had a long history in economics. The continually recurring fluctuating behaviour in economic activity (particularly prices, output, inflation) which has occurred in market economies since the spread of industrialisation and the development of modern financial institutions has drawn the intellectual attention of some of the greatest economic thinkers of the 19th and 20h centuries. The early theories, which are summarised in an excellent manner by Zarnovitz [1985] emphasised either the role of financial factors (in particular the process of credit creation), or the role of the real factors (gestation period and life cycle of capital goods and consumer goods sectors, imbalances between aggregate savings and investment decisions), or the interaction between these two sets of factors. Whatever the balance placed on the various factors just referred to by the early theories, their important characteristic from the perspective of a discussion of nonlinear models of economic dynamics is that they were **endogenous** theories of the business cycle. In modern terminology the business cycle is a self-sustaining oscillation brought about by the lagged reaction of the economic quantities and the nonlinear relations between them.

To these early economic theorists, most of whom were adherents to what Keynes later described as the *"classical school"*, the business cycle posed a deep puzzle in that the economy should be in, or converging towards, a stable general equilibrium. This general equilibrium was often conceived as the Walrasian general equilibrium with its key assumptions of an auctioneer and a central clearing house. A view of the business cycle, alternative to the endogenous cycle theory, which is compatible with a stable general equilibrium grew out of the contribution of Frisch [1933]. According to this view the economy may be modelled mathematically by a linear dynamical system which is stable but which is perturbed continually by exogenous random factors, typically modelled as white noise processes. The development of this concept can be traced to the modern day in the form of the real business cycle theories.

The contrast between the two views of the business cycle, an endogenous stable nonlinear oscillation around an unstable equilibrium point, versus, a randomly perturbed but stable equilibrium point, remains with us to the present day. Both approaches have developed (at non-uniform rates) with the developments in economic theory and econometric testing of underlying economic relationships, on the one hand and the developments in the theories of nonlinear dynamical systems and of randomly perturbed linear dynamical systems, on the other hand.

A similar dichotomy exists in the development of theories of asset price dynamics in financial markets. The early work of Bachelier [1900], who modelled stock price movements as random normal fluctuations, went largely ignored for several decades. The notion that stock price movements had a systematic component was popular in the 1930's, particularly in the form of Dow theory (see e.g. Rhea [1932]) which has developed in the form of modern technical analysis, a tool still widely used by finance industry practitioners, despite the teachings of modern finance theory. Cootner [1960], in discussing the issue of random versus systematic stock price changes, suggested that apparent systematic changes could be explained by his reflecting barriers model. The essential aspect of this model, from the perspective of subsequent developments to be discussed below was that it posited two essentially different classes of investors whose dynamic interaction lead asset prices to follow what we would now term a noisy limit cycle. However the large amount of empirical testing in a variety of financial markets subsequent to the seminal study of Fama [1965] led to a dominance of the view that asset price dynamics are properly modelled as a purely random process. This view is expressed in the form of the efficient markets hypothesis which posits asset price movements as following a random walk. A number of recent empirical studies have begun to cast some doubt on the random walk hypothesis. This literature, which is surveyed by Leroy [1989], has led to a re-emergence in a modern form of the debate between the systematic versus random view on asset price movements.

Many of the early writers on business cycle theory were probably unaware of the work of Poincaré [1899] who in his analysis of celestial mechanics laid the foundations of the modern qualitative-geometric approach to the analysis of nonlinear dynamical systems. Major development of the qualitative theory of dynamical systems occurred in Russia, particularly of the technical tools which have only just begun to be systematically exploited in dynamic economics. Of great importance was the work of Andronov and his co-workers in the 1930's (Andronov et. al. [1966], [1971], [1973]) who developed a range of practical tools for analysing nonlinear dynamical systems. The basic techniques of bifurcation, "catastrophic" changes, and, the method of averaging appear and are systematically employed in these works to analyse two dimensional and occasionally three dimensional nonlinear dynamical systems. Van der Pol [1927] also independently developed some of these concepts and techniques, but the breadth and systematic treatment of the Russian authors remains impressive, even after sixty years. Good use of these techniques is made by Medio [1979] to put many of the early business cycle theories into a consistent nonlinear dynamic framework. Medio's approach is inspired by the works of Goodwin [1950], [1951], [1967] and the earlier works of Kaldor [1940] and Kalecki [1935].

The one concept about which the early dynamicists seem to have been unaware is that of chaos. Poincaré, in his study of nonintegrable Hamiltonian systems was aware of the possible existence of highly complicated recurrent behaviour for systems beyond dimension two. Further insights into these complicated attracting sets were given by Birkhoff [1927] and Cartwright and Littlewood [1945]. However it was only with the brilliant horseshoe construction of Smale [1963] and the computer simulations of Lorenz [1963] that dynamicists came to appreciate the ubiquitous nature and significance of chaotic phenomena.

2. Trends in the Development of Nonlinear Dynamical Techniques

In the introductory section I briefly traced the evolution of the development of the techniques of nonlinear dynamical systems to the time of Smale and Lorenz. The developments since then have been rapid and prodigious, a good overall survey is contained in the two volumes by Jackson [1989]. My aim in this section is to merely highlight those aspects of these developments which are finding, or in my view will find, application to problems in economic dynamics.

The use of the Hopf bifurcation theorem to demonstrate the existence of a limit cycle, as a parameter of interest passes through a critical bifurcation value now seems well established in economics. A number of examples are discussed by Lorenz [1989]. What has been less successfully studied in these applications is the stability of the resulting limit cycle as well as its characteristics. The test for stability of the limit cycle which appears in the literature on the Hopf bifurcation is complex and not always easy to apply when one has a system quantitatively described as in the physical, biological and engineering sciences. It becomes extremely difficult to apply when, as in many economic applications, the differential system is only specified qualitatively. A better way ahead seems to be use of some approximate method. In applying the Hopf bifurcation theory to a problem in population dynamics, Gopalswamy and Aggarwala [1980] found that the method of averaging was a more useful tool for analysing the stability and qualitative features of the limit cycle. This technique has been applied to a range of economic models, by Medio [1979], Chiarella [1990a] and Lux [1991]. The technique, under appropriate conditions, gives a good approximation to the cycle when the bifurcation parameter is close to the critical value. When the bifurcation parameter is not close to the critical value both the Hopf bifurcation theorem and the method of averaging break down. This case is of some relevance in economic dynamic models as it can typically arise when the time constants are vastly different in different markets e.g. asset markets adjusting far more rapidly than goods markets or, when adaptive expectations tend towards the perfect foresight limit.

The kind of cycle that can arise in such situations is of the discontinuous (or relaxation) type, with the economic quantities of interest either undergoing sudden jumps or experiencing sudden changes in direction. There are few considerations of discontinuous cycles in economics, though the catastrophe models of George [1981], Varian [1979] and Zeeman [1974], can be interpreted in this framework. Chiarella [1986] shows that the dynamic instability of monetary dynamics models can be viewed as a manifestation of a discontinuous cycle. Systematic techniques for analysing discontinuous cycles are given by Grasman [1987]. Given the significance of different speeds of adjustment in different markets and changes in the speed of adjustment of expectations in many dynamic economic models of relevance, much remains to be done in the analysis of discontinuous cycles in economics.

Both Hopf bifurcation techniques and discontinuous oscillation techniques usually focus on oscillatory motion around a locally unstable equilibrium point. Such an approach is useful when the (assumed) unique equilibrium point of an economic model becomes locally unstable for a range of parameter values. Frequently the assumption of uniqueness of the equilibrium point is only made for mathematical convenience. However there are many models of interest where it may be more appropriate to consider multiple equilibria, or where the local stability analysis of the assumed unique equilibrium point indicates dynamic behaviour which seems puzzling or perverse as long as the possibility of other equilibrium points is excluded. For example the literature on the dynamics of the government budget restraint or on the dynamics of oligopoly models. The modern theory of nonlinear dynamical systems provides the economic theorist with a range of concepts, in particular homoclinic and heteroclinic orbits and their perturbation, as well as a range of techniques (such as Silnikov's theorem and Melnikov's theorem) that enable him or her to analyse the dynamic picture when the model has multiple equilibrium points or to make better sense out of those models having perverse dynamic behaviour when only one equilibrium point is assumed. The conditions on the underlying economic relationship which give rise to such behaviour in models having non unique equilibrium points need to be fully explored and will be the task of future research in nonlinear economic dynamics.

In earlier discussion I spoke of the dichotomy between the modelling approach which sees economic fluctuations arising as a result of random (white noise) fluctuations on a dynamically stable linear (or topologically equivalent to linear) system, on the one hand, and the approach which sees economic fluctuations arising out of the complicated dynamics of an inherently nonlinear dynamical system (i.e one which is **not** topologically equivalent to a linear system). Whilst I support the thesis that many of the important fluctuations in economic and financial markets arise out of nonlinear dynamic phenomena, this is not to argue that random fluctuation terms are not of significance in a dynamic economic modelling framework which gives a central role to essentially nonlinear dynamic economic mechanisms.

The addition of a white noise term to a nonlinear dynamical system which bifurcates to a limit cycle has a number of important effects. Firstly the critical value of the bifurcation parameter becomes a function of the variance of the white noise term, and secondly the limit cycle becomes a noisy limit cycle with a mean amplitude also a function of the variance of the noise term. If the deterministic version of the model undergoes transition to chaotic motion at some critical parameter value, then that critical value is also affected by the variance of the white noise term. Furthermore the chaotic motion becomes more ordered in that the windows of periodic and quasiperiodic motion tend to disappear. This effect is seen in comparing the Lyapunov exponent of the logistic map with and without the addition of a noise term as discussed for example by Crutchfield, Farmer and Huberman [1982]. Analysis of a nonlinear stochastic dynamical system requires a statistical description, which can be obtained by solving the Fokker-Planck equation for the conditional probability density function. In the chaotic region the effect of the noise term is to smooth out the distribution obtained from the Fokker-Planck equation which would otherwise have the saw-tooth pattern displayed in Kapitaniak [1988]. At a bifurcation point there is a bifurcation of the distribution. To date there have been few analyses of economic models which are both nonlinear and stochastic. Chiarella [1991] analyses a stochastic version of a nonlinear perfect foresight monetary model and shows that the instability of the traditional linear analysis can be interpreted as a bifurcation of the underlying conditional probability density function.

The vast majority of nonlinear dynamic models in economics have been expressible as systems in two or three dimensions. This focus has arisen because it is only such systems that current techniques enable us to analyse relatively easily. Beyond three dimensions computer simulation seems the only tool available and certainly the economic theorist is going to learn to make intelligent use of this tool if he or she is ever to be able to make valid statements about the qualitative effects of policy parameters. Some analytical technical tools are available for higher dimensional systems. For systems having a simple Hopf-bifurcation, centre manifold theory allows us to locate a much lower dimensional manifold (often of dimension two) on which the important dynamics are evolving. There also exists a theory of coupled nonlinear oscillators which would allow the economic theorist to analyse models which have two sectors, for example a real sector and a financial sector each expressed as an oscillatory dynamical system in R^2, but having some coupling. A range of techniques to analyse such systems for both strong coupling and weak coupling are available, see for example Rand and Holmes [1980] and Storti and Rand [1982].

The rapid development of our understanding of nonlinear dynamical phenomena in the natural sciences over the last three decades through the use of a judicious blend of rigorous mathematical theorems, approximation techniques and computer simulation points up a dichotomy of modelling approaches in dynamic economics. The modelling approach discussed by Medio is more akin to that of control engineers in that the mathematical model of the economy is analysed by a range of approximate techniques. More recent applications of nonlinear dynamical concepts in economics have tended to adopt a more formal mathematical approach in that the model under study is modified, or set up initially, so that it satisfies the conditions required by the various theorems available. For example later studies of business cycle models and dynamic IS-LM type models make heavy use of Poincaré-Bendixson theory, however many of these studies require the existence of a bounded compact set at whose boundary the vector field of the model is pointing inwards. It may sometimes be difficult to obtain such a set without some special assumptions, as has been highlighted by Lorenz [1989] and Gabisch and Lorenz [1989]. A similar approach is evident in the analysis of the dynamics of oligopoly models where a wide range of necessary and sufficient conditions, or sometimes just sufficient conditions, are available for analysis of the stability of the equilibrium. To obtain definite results assumptions are placed on marginal cost and marginal revenue functions, which then allow application of powerful theorems from matrix algebra. Whilst our understanding of the dynamics of oligopoly models has been greatly enhanced by such an approach, as evidenced in the recent work of Okuguchi and Szidarovsky [1990], it nevertheless remains the case that once the assumptions are relaxed, oligopoly models are capable of complicated dynamical behaviour. See for example Rand [1978].

The total dynamic picture of nonlinear dynamical models of interest will be better understood by the general modelling approach referred to at the beginning of the previous paragraph. In this task the economic theorist will be aided not only by an array of rigorous theorems but also by the further development of approximate methods and numerical techniques for simulation of the models. It is likely that the new highly sophisticated symbol manipulation computer packages and higher level mathematical programming languages now available will enable the economic theorist to appreciate the dynamic behaviour of quite large (in terms of number of dimensions) models under a wide range of assumptions. Certainly in the physical sciences use of such packages has led to the discovery of results which would not have been possible by reliance on traditional analysis, as discussed for example by Scott and Fee [1991].

3. Potential Development of Some Nonlinear Economic Models

In this section I would like to highlight these areas in economic dynamics which I believe will see some significant advances from an application of the development of the techniques I discussed in the previous section.

The first area that seems to require systematic application of the more recent nonlinear techniques is business cycle theory. In his survey of recent developments of this theory, Zarnovitz [1985] points to a need for a synthesis which systematically exploits substantive uses of nonlinearities and does not give undue prominence to any one of the various endogenous cycle mechanisms that have been proposed over the years. The synthesis should also incorporate elements of the random shock theories. Such a synthesis model would therefore need to have two or three fairly well specified sectors, in particular a real sector, a financial sector and possibly a labour sector. Assuming that each sector is at least of dimension 2 then the minimum dimension of the synthesised model would be at least 6. Furthermore the addition of the stochastic shock term would make the model both nonlinear and stochastic. A range of models for each sector are well established see e.g. Medio [1979] for models of the real sector, Zhang [1991] for models of the financial sector, and Ferri and Greenberg [1989] for models of the labour market. The developing nonlinear models in conjunction with the symbolic and numerical computer packages becoming available will enable the economic theorist to analyse the synthesis model combing these various sectors. Such an analysis will proceed via an analysis of the equilibrium points of the model, their regions of local stability and instability, analysis of bifurcation points and transitions to chaos, analysis of homoclinic and heteroclinic orbits, approximate analysis of the time trajectories of key economic quantities at these critical points, use of centre manifold theory, the method of averaging and the techniques of coupled oscillators with a view to obtaining an understanding of how the models fluctuations relate to underlying economic quantities such as monetary and fiscal parameters or labour market structure. Numerical simulations of such models with a view to understanding their time series properties may be of some use in analysing time series of real economic data and determining whether certain model configurations conform to some of the stylised facts of business cycle theory. A range of synthesis models will be obtained depending upon which factors in the various sectors are emphasised. e.g. the financial sector might be specified according to neoclassical monetary theory as in Zhang [1991] or according to Minsky type models as in Semmler [1986].

Another area of dynamic economic analysis which seems to require a nonlinear stochastic analysis is that of exchange rate dynamics. It is now well documented in empirical literature that exchange rate models based on a variety of traditional modelling frameworks (e.g. monetary, portfolio balance, representative agent intertemporal utility maximising) perform quite poorly when tested against actual exchange rate time series, even in ex-post historical simulations. The aforementioned models are of the stable linear stochastic shock variety. In order to retain the stable linear aspect of this modelling framework some recent research has focused on a search for linear long-run relationships amongst exchange rates and factors determining them (this is the so-called 'cointegration' theory) or allowing the second moment of the stochastic shock terms to be nonlinear, leading to ARCH models as discussed by Diebold [1988]. Models which are nonlinear in the underlying economic relationship and capable of generating complicated dynamic behaviour provide an alternative paradigm which may prove to be more fruitful in explaining the observed time series properties of exchange rates, particularly their apparent excess volatility. A number of models of exchange rate dynamics which are nonlinear (or can be interpreted as nonlinear) have begun to appear, in particular Frankel and Froot [1986], Krugman [1988], Froot and Obstfeld [1989] and Chiarella [1990b]. The latter author shows that the nonlinear framework has the additional advantage that it may be used to overcome the saddle point instability behaviour of many of the standard exchange rate models, a feature which has been considered unsatisfactory by many authors. The recent book of Rosser [1991] contains a good discussion of exchange rate dynamics in the context of nonlinear dynamics. Empirical evidence of nonlinear behaviour in exchange rate time series has been given by Prince [1987], who investigates the Lyapunov exponents of various exchange rate time series, and Kräger and Kugler [1991] who fit SETAR type nonlinear models to a range of exchange rates. Much more empirical testing of nonlinear models, as well as comparison with the alternative ARCH paradigm, needs to be done.

Finally, the area of asset price dynamics would seem ripe to benefit from analysis from a nonlinear dynamics perspective. In the first section I mentioned the recent empirical studies which cast some doubt on the efficient markets / random walk paradigm. In that section I also mentioned the contrast between theories of asset price dynamics that on the one hand view the fluctuations in asset prices as being essentially due to exogenous stochastic influences, and on the other hand see such fluctuations as arising essentially from some underlying systematic cause. In the latter class of model price fluctuations can be shown to arise from some nonlinear dynamic mechanism. The application of techniques of nonlinear dynamics will allow us to explore more deeply such models, though the result will almost certainly not be the technical analyst's chimerical goal of a formula which will allow perfect prediction of market movements. We have already discussed Cootner's reflecting barriers model which was never fully developed probably because it was formulated at about the beginning of the period of dominance of the efficient markets / random walk paradigm. However Cootner's basic insight that the systematic component of asset price movements arises as a result of the dynamic interaction of two fundamentally different classes of investors continued to appear in what was viewed as a kind of counter culture literature during the 70's and 80's. Two important contributions whose significance was not widely appreciated at the time of publication were those of Zeeman [1974] and Beja and Goldman [1980]. The essential feature of the models of these authors is that they posit two essentially different groups of investors, one group (the fundamentalists) basing investment decisions on fundamental factors, à la the traditional Walrasian paradigm, the second group (the chartists) basing investment decisions on their expectation of price trends. The dynamic interaction of these two groups (via price adjustments due to excess demand pressure and dynamic formation of chartists expectations) can be expressed as a system of nonlinear differential equations. This same basic idea can be found in Genotte and Leland [1990] and Frankel and Froot [1986]. Chiarella [1992] shows that this type of model is capable of a wide range of dynamic behaviour including convergence to a stable, white noise perturbed, limit point (in conformity with the dominant / efficient markets paradigm), noisy limit cycles and noisy chaotic motion. The models which are typically the expression of the efficient / random-walk paradigm seem implicitly to be capable of generating only the first of these types of dynamic behaviour. A conjecture worth exploring is that many of the puzzles in recent empirical studies of asset price dynamics may stem from the failure of the models underpinning the efficient markets / random-walk paradigm to allow a sufficiently rich set of dynamic possibilities. Exploration of this conjecture will proceed in two directions. Firstly, further elaboration of the nonlinear dynamic behaviour of models of asset markets microstructure so as to give some theoretical guidance as to the type of theoretical models empiricists should be seeking. Secondly, the development of empirical techniques appropriate for the testing of models of asset price dynamics which allow the full range of nonlinear dynamic possibilities. Such a development is likely to grow out of the nonlinear time series techniques elaborated by Tong [1990].

4. Conclusions

I have briefly sketched the interaction of the evolution of developments in nonlinear dynamical systems theory and nonlinear economic dynamics. The developments, over the last two decades, of techniques for the analysis of nonlinear dynamical systems as well as nonlinear time series analysis should enable economists to view more closely, at both a theoretical and empirical level, the very rich dynamics of models which hitherto have only been viewed through the long distance telescope of local linear analysis. Ichimura [1955], in surveying the developments in nonlinear economic dynamics at his time of writing, entitled his article "Towards a general nonlinear macrodynamic theory of economic fluctuations". With the recent developments the economic theorist has entered the world of such a general theory and in the coming decades the techniques and models I have discussed in the previous sections should help him or her to explore it more fully.

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